Prime Number Theorem And ...

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Chapter 1

The project

The project github page is `https://github.com/AlexKontorovich/PrimeNumberTheoremAnd`.

The project docs page is `https://alexkontorovich.github.io/PrimeNumberTheoremAnd/docs`.

The first main goal is to prove the Prime Number Theorem in Lean. (This remains one of the outstanding problems on Wiedijk's list of 100 theorems to formalize.) Note that PNT has been formalized before, first by Avigad et al in Isabelle, `https://arxiv.org/abs/cs/0509025` following the Selberg / Erdos method, then by Harrison in HOL Light `https://www.cl.cam.ac.uk/$\sim$jrh13/papers/mikefest.html` via Newman's proof. Carniero gave another formalization in Metamath of the Selberg / Erdos method: `https://arxiv.org/abs/1608.02029`, and Eberl-Paulson gave a formalization of Newman's proof in Isabelle: `https://www.isa-afp.org/entries/Prime_Number_Theorem.html`

Continuations of this project aim to extend this work to primes in progressions (Dirichlet's theorem), Chebotarev's density theorem, etc etc.

There are (at least) three approaches to PNT that we may want to pursue simultaneously. The quickest, at this stage, is likely to follow the "Euler Products" project by Michael Stoll, which has a proof of PNT missing only the Wiener-Ikehara Tauberian theorem.

The second develops some complex analysis (residue calculus on rectangles, argument principle, Mellin transforms), to pull contours and derive a PNT with an error term which is stronger than any power of log savings.

The third approach, which will be the most general of the three, is to: (1) develop the residue calculus et al, as above, (2) add the Hadamard factorization theorem, (3) use it to prove the zero-free region for zeta via Hoffstein-Lockhart+Goldfeld-Hoffstein-Liemann (which generalizes to higher degree L-functions), and (4) use this to prove the prime number theorem with exp-root-log savings.

A word about the expected "rate-limiting-steps" in each of the approaches.

(*) In approach (1), I think it will be the fact that the Fourier transform is a bijection on the Schwartz class. There is a recent PR (`https://github.com/leanprover-community/mathlib4/pull/9773`) with David Loeffler and Heather Macbeth making the first steps in that direction, just computing the (Frechet) derivative of the Fourier transform. One will need to iterate on that to get arbitrary derivatives, to conclude that the transform of a Schwartz function is Schwartz. Then to get the bijection, we need an inversion formula. We can derive Fourier inversion *from* Mellin inversion! So it seems that the most important thing to start is Perron's formula.

(*) In approach (2), there are two rate-limiting-steps, neither too serious (in my estimation). The first is how to handle meromorphic functions on rectangles. Perhaps in this project, it should not be done in any generality, but on a case by case basis. There are two simple poles whose residues need to be computed in the proof of the Perron formula, and one simple pole in the log-derivative of zeta, nothing too complicated, and maybe we shouldn't get bogged down in the general case. The other is the fact that the ϵ-smoothed Chebyshev function differs from the unsmoothed by ϵX (and not $\epsilon X \log X$, as follows from a trivial bound). This needs a Brun-Titchmarsh type theorem, perhaps can be done even more easily in this case with a Selberg sieve, on which there is (partial?) progress in Mathlib.

(*) In approach (3), it's obviously the Hadamard factorization, which needs quite a lot of nontrivial mathematics. (But after that, the math is not hard, on top of things in approach (2) – and if we're getting the strong error term, we can afford to lose $\log X$ in the Chebyshev discussion above...).

Chapter 2

First approach: Wiener-Ikehara Tauberian theorem

2.1 A Fourier-analytic proof of the Wiener-Ikehara theorem

The Fourier transform of an absolutely integrable function $\psi : \mathbb{R} \to \mathbb{C}$ is defined by the formula

$$\hat{\psi}(u) := \int_{\mathbb{R}} e(-tu)\psi(t)\ dt$$

where $e(\theta) := e^{2\pi i\theta}$.

Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetic function such that $\sum_{n=1}^{\infty} \frac{|f(n)|}{n^\sigma} < \infty$ for all $\sigma > 1$. Then the Dirichlet series

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

is absolutely convergent for $\sigma > 1$.

Lemma 2.1.1 (first-fourier)**.** If $\psi : \mathbb{R} \to \mathbb{C}$ is integrable and $x > 0$, then for any $\sigma > 1$

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^\sigma} \hat{\psi}(\frac{1}{2\pi} \log \frac{n}{x}) = \int_{\mathbb{R}} F(\sigma + it)\psi(t)x^{it}\ dt.$$

Proof. By the definition of the Fourier transform, the left-hand side expands as

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} \frac{f(n)}{n^\sigma} \psi(t) e(-\frac{1}{2\pi} t \log \frac{n}{x})\ dt$$

while the right-hand side expands as

$$\int_{\mathbb{R}} \sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma+it}} \psi(t) x^{it}\ dt.$$

Since

$$\frac{f(n)}{n^\sigma} \psi(t) e(-\frac{1}{2\pi} t \log \frac{n}{x}) = \frac{f(n)}{n^{\sigma+it}} \psi(t) x^{it}$$

the claim then follows from Fubini's theorem. □

Lemma 2.1.2 (second-fourier)**.** If $\psi : \mathbb{R} \to \mathbb{C}$ is continuous and compactly supported and $x > 0$, then for any $\sigma > 1$

$$\int_{-\log x}^{\infty} e^{-u(\sigma-1)} \hat{\psi}(\frac{u}{2\pi})\ du = x^{\sigma-1} \int_{\mathbb{R}} \frac{1}{\sigma+it-1} \psi(t) x^{it}\ dt.$$

Proof. The left-hand side expands as

$$\int_{-\log x}^{\infty} \int_{\mathbb{R}} e^{-u(\sigma-1)} \psi(t) e(-\tfrac{tu}{2\pi})\ dt\ du$$
$$? = x^{\sigma-1} \int_{\mathbb{R}} \tfrac{1}{\sigma+it-1} \psi(t) x^{it}\ dt$$

so by Fubini's theorem it suffices to verify the identity

$$\begin{aligned}
\int_{-\log x}^{\infty} e^{-u(\sigma-1)} e(-\frac{tu}{2\pi})\ du &= \int_{-\log x}^{\infty} e^{(it-\sigma+1)u}\ du \\
&= \frac{1}{it-\sigma+1} e^{(it-\sigma+1)u}\ \Big|_{-\log x}^{\infty} \\
&= x^{\sigma-1} \frac{1}{\sigma+it-1} x^{it}
\end{aligned}$$

□

Now let $A \in \mathbb{C}$, and suppose that there is a continuous function $G(s)$ defined on $\mathrm{Re} s \geq 1$ such that $G(s) = F(s) - \frac{A}{s-1}$ whenever $\mathrm{Re} s > 1$. We also make the Chebyshev-type hypothesis

$$\sum_{n \leq x} |f(n)| \ll x \qquad (2.1)$$

for all $x \geq 1$ (this hypothesis is not strictly necessary, but simplifies the arguments and can be obtained fairly easily in applications).

Lemma 2.1.3 (Preliminary decay bound I)**.** If $\psi : \mathbb{R} \to \mathbb{C}$ is absolutely integrable then

$$|\hat{\psi}(u)| \leq \|\psi\|_1$$

for all $u \in \mathbb{R}$. where C is an absolute constant.

Proof. Immediate from the triangle inequality. □

Lemma 2.1.4 (Preliminary decay bound II)**.** If $\psi : \mathbb{R} \to \mathbb{C}$ is absolutely integrable and of bounded variation, then

$$|\hat{\psi}(u)| \leq \|\psi\|_{TV} / 2\pi |u|$$

for all non-zero $u \in \mathbb{R}$.

Proof. By Lebesgue–Stiejtes integration by parts we have

$$2\pi i u \hat{\psi}(u) = \int_{\mathbb{R}} e(-tu) d\psi(t)$$

and the claim then follows from the triangle inequality. □

Lemma 2.1.5 (Preliminary decay bound III)**.** If $\psi : \mathbb{R} \to \mathbb{C}$ is absolutely integrable, absolutely continuous, and ψ' is of bounded variation, then

$$|\hat{\psi}(u)| \leq \|\psi'\|_{TV}/(2\pi|u|)^2$$

for all non-zero $u \in \mathbb{R}$.

Proof. Should follow from previous lemma. □

Lemma 2.1.6 (Decay bound, alternate form)**.** If $\psi : \mathbb{R} \to \mathbb{C}$ is absolutely integrable, absolutely continuous, and ψ' is of bounded variation, then

$$|\hat{\psi}(u)| \leq (\|\psi\|_1 + \|\psi'\|_{TV}/(2\pi)^2)/(1+|u|^2)$$

for all $u \in \mathbb{R}$.

Proof. Should follow from previous lemmas. □

Lemma 2.1.7 (Decay bounds)**.** If $\psi : \mathbb{R} \to \mathbb{C}$ is C^2 and obeys the bounds

$$|\psi(t)|, |\psi''(t)| \leq A/(1+|t|^2)$$

for all $t \in \mathbb{R}$, then

$$|\hat{\psi}(u)| \leq CA/(1+|u|^2)$$

for all $u \in \mathbb{R}$, where C is an absolute constant.

Proof. From two integration by parts we obtain the identity

$$(1+u^2)\hat{\psi}(u) = \int_{\mathbf{R}} (\psi(t) - \frac{u}{4\pi^2}\psi''(t))e(-tu)\ dt.$$

Now apply the triangle inequality and the identity $\int_{\mathbf{R}} \frac{dt}{1+t^2}\ dt = \pi$ to obtain the claim with $C = \pi + 1/4\pi$. □

Lemma 2.1.8 (Limiting Fourier identity)**.** If $\psi : \mathbb{R} \to \mathbb{C}$ is C^2 and compactly supported and $x \geq 1$, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}\hat{\psi}(\frac{1}{2\pi}\log\frac{n}{x}) - A\int_{-\log x}^{\infty} \hat{\psi}(\frac{u}{2\pi})\ du = \int_{\mathbb{R}} G(1+it)\psi(t)x^{it}\ dt.$$

Proof. By Lemma 2.1.1 and Lemma 2.1.2, we know that for any $\sigma > 1$, we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^\sigma}\hat{\psi}(\frac{1}{2\pi}\log\frac{n}{x}) - Ax^{1-\sigma}\int_{-\log x}^{\infty} e^{-u(\sigma-1)}\hat{\psi}(\frac{u}{2\pi})\ du = \int_{\mathbb{R}} G(\sigma+it)\psi(t)x^{it}\ dt.$$

Now take limits as $\sigma \to 1$ using dominated convergence together with (2.1) and Lemma 2.1.7 to obtain the result. □

Corollary 2.1.1 (Corollary of limiting identity)**.** With the hypotheses as above, we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}\hat{\psi}(\frac{1}{2\pi}\log\frac{n}{x}) = A\int_{-\infty}^{\infty} \hat{\psi}(\frac{u}{2\pi})\ du + o(1)$$

as $x \to \infty$.

Proof. Immediate from the Riemann-Lebesgue lemma, and also noting that $\int_{-\infty}^{-\log x} \hat{\psi}(\frac{u}{2\pi})\, du = o(1)$. □

Lemma 2.1.9 (Smooth Urysohn lemma)**.** If I is a closed interval contained in an open interval J, then there exists a smooth function $\Psi : \mathbb{R} \to \mathbb{R}$ with $1_I \leq \Psi \leq 1_J$.

Proof. A standard analysis lemma, which can be proven by convolving 1_K with a smooth approximation to the identity for some interval K between I and J. Note that we have "SmoothBumpFunction"s on smooth manifolds in Mathlib, so this shouldn't be too hard... □

Lemma 2.1.10 (Limiting identity for Schwartz functions)**.** The previous corollary also holds for functions ψ that are assumed to be in the Schwartz class, as opposed to being C^2 and compactly supported.

Proof. For any $R > 1$, one can use a smooth cutoff function (provided by Lemma 2.1.9 to write $\psi = \psi_{\leq R} + \psi_{>R}$, where $\psi_{\leq R}$ is C^2 (in fact smooth) and compactly supported (on $[-R, R]$), and $\psi_{>R}$ obeys bounds of the form

$$|\psi_{>R}(t)|, |\psi''_{>R}(t)| \ll R^{-1}/(1 + |t|^2)$$

where the implied constants depend on ψ. By Lemma 2.1.7 we then have

$$\hat{\psi}_{>R}(u) \ll R^{-1}/(1 + |u|^2).$$

Using this and (2.1) one can show that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \hat{\psi}_{>R}(\frac{1}{2\pi} \log \frac{n}{x}), A \int_{-\infty}^{\infty} \hat{\psi}_{>R}(\frac{u}{2\pi})\, du \ll R^{-1}$$

(with implied constants also depending on A), while from Lemma 2.1.1 one has

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \hat{\psi}_{\leq R}(\frac{1}{2\pi} \log \frac{n}{x}) = A \int_{-\infty}^{\infty} \hat{\psi}_{\leq R}(\frac{u}{2\pi})\, du + o(1).$$

Combining the two estimates and letting R be large, we obtain the claim. □

Lemma 2.1.11 (Bijectivity of Fourier transform)**.** The Fourier transform is a bijection on the Schwartz class. [Note: only surjectivity is actually used.]

Proof. This is a standard result in Fourier analysis. It can be proved here by appealing to Mellin inversion, Theorem **??**. In particular, given f in the Schwartz class, let $F : \mathbb{R}_+ \to \mathbb{C} : x \mapsto f(\log x)$ be a function in the "Mellin space"; then the Mellin transform of F on the imaginary axis $s = it$ is the Fourier transform of f. The Mellin inversion theorem gives Fourier inversion. □

Corollary 2.1.2 (Smoothed Wiener-Ikehara)**.** If $\Psi : (0, \infty) \to \mathbb{C}$ is smooth and compactly supported away from the origin, then,

$$\sum_{n=1}^{\infty} f(n) \Psi(\frac{n}{x}) = Ax \int_{0}^{\infty} \Psi(y)\, dy + o(x)$$

as $x \to \infty$.

Proof. By Lemma 2.1.11, we can write

$$y\Psi(y) = \hat{\psi}(\frac{1}{2\pi} \log y)$$

for all $y > 0$ and some Schwartz function ψ. Making this substitution, the claim is then equivalent after standard manipulations to

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \hat{\psi}(\frac{1}{2\pi} \log \frac{n}{x}) = A \int_{-\infty}^{\infty} \hat{\psi}(\frac{u}{2\pi})\ du + o(1)$$

and the claim follows from Lemma 2.1.10. □

Now we add the hypothesis that $f(n) \geq 0$ for all n.

Proposition 2.1.1 (Wiener-Ikehara in an interval)**.** For any closed interval $I \subset (0, +\infty)$, we have

$$\sum_{n=1}^{\infty} f(n) 1_I(\frac{n}{x}) = Ax|I| + o(x).$$

Proof. Use Lemma 2.1.9 to bound 1_I above and below by smooth compactly supported functions whose integral is close to the measure of $|I|$, and use the non-negativity of f. □

Corollary 2.1.3 (Wiener-Ikehara Theorem (1))**.** We have

$$\sum_{n \leq x} f(n) = Ax + o(x).$$

Proof. Apply the preceding proposition with $I = [\varepsilon, 1]$ and then send ε to zero (using (2.1) to control the error). □

2.2 Weak PNT

Theorem 2.2.1 (WeakPNT)**.** We have

$$\sum_{n \leq x} \Lambda(n) = x + o(x).$$

Proof. Already done by Stoll, assuming Wiener-Ikehara. □

2.3 Removing the Chebyshev hypothesis

In this section we do *not* assume the bound (2.1), but instead derive it from the other hypotheses.

Lemma 2.3.1 (limiting-fourier-variant)**.** If $\psi : \mathbb{R} \to \mathbb{C}$ is C^2 and compactly supported with f and $\hat{\psi}$ non-negative, and $0 < x$, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \hat{\psi}(\frac{1}{2\pi} \log \frac{n}{x}) - A \int_{-\log x}^{\infty} \hat{\psi}(\frac{u}{2\pi})\ du = \int_{\mathbb{R}} G(1 + it)\psi(t)x^{it}\ dt.$$

Proof. Repeat the proof of Lemma 2.3.1, but use monotone convergence instead of dominated convergence. (The proof should be simpler, as one no longer needs to establish domination for the sum.) □

Corollary 2.3.1 (crude-upper-bound)**.** If $\psi : \mathbb{R} \to \mathbb{C}$ is C^2 and compactly supported with f and $\hat{\psi}$ non-negative, then there exists a constant B such that

$$|\sum_{n=1}^{\infty} \frac{f(n)}{n} \hat{\psi}(\frac{1}{2\pi} \log \frac{n}{x})| \leq B$$

for all $x > 0$.

Proof. This readily follows from the previous lemma and the triangle inequality. □

Corollary 2.3.2 (auto-cheby)**.** One has

$$\sum_{n \leq x} f(n) = O(x)$$

for all $x \geq 1$.

Proof. By applying Corollary 2.3.1 for a specific compactly supported function ψ, one can obtain a bound of the form $\sum_{(1-\varepsilon)x<n\leq x} f(n) = O(x)$ for all x and some absolute constant ε (which can be made explicit).

If C is a sufficiently large constant, the claim $|\sum_{n\leq x} f(n)| \leq Cx$ can now be proven by strong induction on x, as the claim for $(1-\varepsilon)x$ implies the claim for x by the triangle inequality (and the claim is trivial for $x < 1$). □

Theorem 2.3.1 (Wiener-Ikehara Theorem (2))**.** We have

$$\sum_{n \leq x} f(n) = Ax + o(x).$$

Proof. Use Corollary 2.3.2 to remove the Chebyshev hypothesis in Theorem 2.1.3. □

2.4 The prime number theorem in arithmetic progressions

Lemma 2.4.1 (WeakPNT-character)**.** If $q \geq 1$ and a is coprime to q, and $\mathrm{Re} s > 1$, we have

$$\sum_{n:n=a\ (q)} \frac{\Lambda(n)}{n^s} = -\frac{1}{\varphi(q)} \sum_{\chi\ (q)} \overline{\chi(a)} \frac{L'(s,\chi)}{L(s,\chi)}.$$

Proof. From the Fourier inversion formula on the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$, we have

$$1_{n=a\ (q)} = \frac{\varphi(q)}{q} \sum_{\chi\ (q)} \overline{\chi(a)}\chi(n).$$

On the other hand, from standard facts about L-series we have for each character χ that

$$\sum_{n} \frac{\Lambda(n)\chi(n)}{n^s} = -\frac{L'(s,\chi)}{L(s,\chi)}.$$

Combining these two facts, we obtain the claim. □

Proposition 2.4.1 (WeakPNT-AP-prelim)**.** If $q \geq 1$ and a is coprime to q, the Dirichlet series $\sum_{n \leq x: n = a\ (q)} \frac{\Lambda(n)}{n^s}$ converges for $\mathrm{Re}(s) > 1$ to $\frac{1}{\varphi(q)} \frac{1}{s-1} + G(s)$ where G has a continuous extension to $\mathrm{Re}(s) = 1$.

Proof. We expand out the left-hand side using Lemma 2.4.1. The contribution of the non-principal characters χ extend continuously to $\mathrm{Re}(s) = 1$ thanks to the non-vanishing of $L(s, \chi)$ on this line (which should follow from another component of this project), so it suffices to show that for the principal character χ_0, that

$$-\frac{L'(s, \chi_0)}{L(s, \chi_0)} - \frac{1}{s-1}$$

also extends continuously here. But we already know that

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$$

extends, and from Euler product machinery one has the identity

$$\frac{L'(s, \chi_0)}{L(s, \chi_0)} = \frac{\zeta'(s)}{\zeta(s)} + \sum_{p|q} \frac{\log p}{p^s - 1}.$$

Since there are only finitely many primes dividing q, and each summand $\frac{\log p}{p^s-1}$ extends continuously, the claim follows. □

Theorem 2.4.1 (WeakPNT-AP)**.** If $q \geq 1$ and a is coprime to q, we have

$$\sum_{n \leq x: n = a\ (q)} \Lambda(n) = \frac{x}{\varphi(q)} + o(x).$$

Proof. Apply Theorem 2.1.3 (or Theorem 2.3.1 to avoid checking the Chebyshev condition) using Proposition 2.4.1. □

2.5 The Chebotarev density theorem: the case of cyclotomic extensions

In this section, K is a number field, $L = K(\mu_m)$ for some natural number m, and $G = Gal(K/L)$.

The goal here is to prove the Chebotarev density theorem for the case of cyclotomic extensions.

Lemma 2.5.1 (Dedekind-factor)**.** We have

$$\zeta_L(s) = \prod_\chi L(\chi, s)$$

for $\Re(s) > 1$, where χ runs over homomorphisms from G to $\mathbb{C}^\times$ and L is the Artin L-function.

Proof. See Propositions 7.1.16, 7.1.19 of `https://www.math.ucla.edu/~sharifi/algnum.pdf`. □

Lemma 2.5.2 (Simple pole)**.** ζ_L has a simple pole at $s = 1$.

Proof. See Theorem 7.1.12 of `https://www.math.ucla.edu/~sharifi/algnum.pdf`. ☐

Lemma 2.5.3 (Dedekind-nonvanishing)**.** For any non-principal character χ of $Gal(K/L)$, $L(\chi, s)$ does not vanish for $\Re(s) = 1$.

Proof. For $s = 1$, this will follow from Lemmas 2.5.1, 2.5.2. For the rest of the line, one should be able to adapt the arguments for the Dirichet L-function. ☐

2.6 The Chebotarev density theorem: the case of abelian extensions

(Use the arguments in Theorem 7.2.2 of `https://www.math.ucla.edu/~sharifi/algnum.pdf` to extend the previous results to abelian extensions (actually just cyclic extensions would suffice))

2.7 The Chebotarev density theorem: the general case

(Use the arguments in Theorem 7.2.2 of `https://www.math.ucla.edu/~sharifi/algnum.pdf` to extend the previous results to arbitrary extensions

Lemma 2.7.1 (PNT for one character)**.** For any non-principal character χ of $Gal(K/L)$,

$$\sum_{N\mathfrak{p} \le x} \chi(\mathfrak{p}) \log N\mathfrak{p} = o(x).$$

Proof. This should follow from Lemma 2.5.3 and the arguments for the Dirichlet L-function. (It may be more convenient to work with a von Mangoldt type function instead of $\log N\mathfrak{p}$). ☐

Chapter 3

Second approach

3.1 Residue calculus on rectangles

This files gathers definitions and basic properties about rectangles.

The border of a rectangle is the union of its four sides.

Definition 3.1.1 (RectangleBorder)**.** A Rectangle's border, given corners z and w is the union of the four sides.

Definition 3.1.2 (RectangleIntegral)**.** A RectangleIntegral of a function f is one over a rectangle determined by z and w in $\mathbb{C}$. We will sometimes denote it by $\int_z^w f$. (There is also a primed version, which is $1/(2\pi i)$ times the original.)

Definition 3.1.3 (UpperUIntegral)**.** An UpperUIntegral of a function f comes from $\sigma + i\infty$ down to $\sigma + iT$, over to $\sigma' + iT$, and back up to $\sigma' + i\infty$.

Definition 3.1.4 (LowerUIntegral)**.** A LowerUIntegral of a function f comes from $\sigma - i\infty$ up to $\sigma - iT$, over to $\sigma' - iT$, and back down to $\sigma' - i\infty$.

It is very convenient to define integrals along vertical lines in the complex plane, as follows.

Definition 3.1.5 (VerticalIntegral)**.** Let f be a function from $\mathbb{C}$ to $\mathbb{C}$, and let σ be a real number. Then we define

$$\int_{(\sigma)} f(s)ds = \int_{\sigma - i\infty}^{\sigma + i\infty} f(s)ds.$$

We also have a version with a factor of $1/(2\pi i)$.

Lemma 3.1.1 (DiffVertRect-eq-UpperLowerUs)**.** The difference of two vertical integrals and a rectangle is the difference of an upper and a lower U integrals.

Proof. Follows directly from the definitions. □

Theorem 3.1.1 (existsDifferentiableOn-of-bddAbove)**.** If f is differentiable on a set s except at $c \in s$, and f is bounded above on $s \setminus \{c\}$, then there exists a differentiable function g on s such that f and g agree on $s \setminus \{c\}$.

Proof. This is the Riemann Removable Singularity Theorem, slightly rephrased from what's in Mathlib. (We don't care what the function g is, just that it's holomorphic.) □

Theorem 3.1.2 (HolomorphicOn.vanishesOnRectangle)**.** If f is holomorphic on a rectangle z and w, then the integral of f over the rectangle with corners z and w is 0.

Proof. This is in a Mathlib PR. □

The next lemma allows to zoom a big rectangle down to a small square, centered at a pole.

Lemma 3.1.2 (RectanglePullToNhdOfPole)**.** If f is holomorphic on a rectangle z and w except at a point p, then the integral of f over the rectangle with corners z and w is the same as the integral of f over a small square centered at p.

Proof. Chop the big rectangle with two vertical cuts and two horizontal cuts into smaller rectangles, the middle one being the desired square. The integral over each of the outer rectangles vanishes, since f is holomorphic there. (The constant c being "small enough" here just means that the inner square is strictly contained in the big rectangle.) □

Lemma 3.1.3 (ResidueTheoremAtOrigin)**.** The rectangle (square) integral of $f(s) = 1/s$ with corners $-1 - i$ and $1 + i$ is equal to $2\pi i$.

Proof. This is a special case of the more general result above. □

Lemma 3.1.4 (ResidueTheoremOnRectangleWithSimplePole)**.** Suppose that f is a holomorphic function on a rectangle, except for a simple pole at p. By the latter, we mean that there is a function g holomorphic on the rectangle such that, $f = g + A/(s - p)$ for some $A \in \mathbb{C}$. Then the integral of f over the rectangle is A.

Proof. Replace f with $g + A/(s - p)$ in the integral. The integral of g vanishes by Lemma 3.1.2. To evaluate the integral of $1/(s - p)$, pull everything to a square about the origin using Lemma 3.1.2, and rescale by c; what remains is handled by Lemma 3.1.3. □

3.2 Perron Formula

In this section, we prove the Perron formula, which plays a key role in our proof of Mellin inversion.

The following is preparatory material used in the proof of the Perron formula, see Lemma 3.2.16.

Lemma 3.2.1 (zeroTendstoDiff)**.** If the limit of 0 is $L_1 - L_2$, then $L_1 = L_2$.

Proof. Obvious. □

Lemma 3.2.2 (RectangleIntegral-tendsTo-VerticalIntegral)**.** Let $\sigma, \sigma' \in \mathbb{R}$, and $f : \mathbb{C} \to \mathbb{C}$ such that the vertical integrals $\int_{(\sigma)} f(s)ds$ and $\int_{(\sigma')} f(s)ds$ exist and the horizontal integral $\int_{(\sigma)}^{\sigma'} f(x + yi)dx$ vanishes as $y \to \pm\infty$. Then the limit of rectangle integrals

$$\lim_{T\to\infty} \int_{\sigma - iT}^{\sigma' + iT} f(s)ds = \int_{(\sigma')} f(s)ds - \int_{(\sigma)} f(s)ds.$$

Proof. Almost by definition. □

Lemma 3.2.3 (RectangleIntegral-tendsTo-UpperU)**.** Let $\sigma, \sigma' \in \mathbb{R}$, and $f : \mathbb{C} \to \mathbb{C}$ such that the vertical integrals $\int_{(\sigma)} f(s)ds$ and $\int_{(\sigma')} f(s)ds$ exist and the horizontal integral $\int_{(\sigma)}^{\sigma'} f(x+yi)dx$ vanishes as $y \to \pm\infty$. Then the limit of rectangle integrals

$$\int_{\sigma+iT}^{\sigma'+iU} f(s)ds$$

as $U \to \infty$ is the "UpperUIntegral" of f.

Proof. Almost by definition. □

Lemma 3.2.4 (RectangleIntegral-tendsTo-LowerU)**.** Let $\sigma, \sigma' \in \mathbb{R}$, and $f : \mathbb{C} \to \mathbb{C}$ such that the vertical integrals $\int_{(\sigma)} f(s)ds$ and $\int_{(\sigma')} f(s)ds$ exist and the horizontal integral $\int_{(\sigma)}^{\sigma'} f(x+yi)dx$ vanishes as $y \to -\infty$. Then the limit of rectangle integrals

$$\int_{\sigma-iU}^{\sigma'-iT} f(s)ds$$

as $U \to \infty$ is the "LowerUIntegral" of f.

Proof. Almost by definition. □

TODO : Move to general section

Lemma 3.2.5 (limitOfConstant)**.** Let $a : \mathbb{R} \to \mathbb{C}$ be a function, and let $\sigma > 0$ be a real number. Suppose that, for all $\sigma, \sigma' > 0$, we have $a(\sigma') = a(\sigma)$, and that $\lim_{\sigma\to\infty} a(\sigma) = 0$. Then $a(\sigma) = 0$.

Proof. □

Lemma 3.2.6 (limitOfConstantLeft)**.** Let $a : \mathbb{R} \to \mathbb{C}$ be a function, and let $\sigma < -3/2$ be a real number. Suppose that, for all $\sigma, \sigma' > 0$, we have $a(\sigma') = a(\sigma)$, and that $\lim_{\sigma\to-\infty} a(\sigma) = 0$. Then $a(\sigma) = 0$.

Proof. □

Lemma 3.2.7 (tendsto-rpow-atTop-nhds-zero-of-norm-lt-one)**.** Let $x > 0$ and $x < 1$. Then

$$\lim_{\sigma\to\infty} x^\sigma = 0.$$

Proof. Standard. □

Lemma 3.2.8 (tendsto-rpow-atTop-nhds-zero-of-norm-gt-one)**.** Let $x > 1$. Then

$$\lim_{\sigma\to-\infty} x^\sigma = 0.$$

Proof. Standard. □

Lemma 3.2.9 (isHolomorphicOn)**.** Let $x > 0$. Then the function $f(s) = x^s/(s(s+1))$ is holomorphic on the half-plane $\{s \in \mathbb{C} : \Re(s) > 0\}$.

Proof. □

Lemma 3.2.10 (integralPosAux)**.** The integral

$$\int_{\mathbb{R}} \frac{1}{|(1+t^2)(2+t^2)|^{1/2}} dt$$

is positive (and hence convergent - since a divergent integral is zero in Lean, by definition).

Proof. □

Lemma 3.2.11 (vertIntBound)**.** Let $x > 0$ and $\sigma > 1$. Then

$$\left| \int_{(\sigma)} \frac{x^s}{s(s+1)} ds \right| \leq x^\sigma \int_{\mathbb{R}} \frac{1}{|(1+t^2)(2+t^2)|^{1/2}} dt.$$

Proof. Triangle inequality and pointwise estimate. □

Lemma 3.2.12 (vertIntBoundLeft)**.** Let $x > 1$ and $\sigma < -3/2$. Then

$$\left| \int_{(\sigma)} \frac{x^s}{s(s+1)} ds \right| \leq x^\sigma \int_{\mathbb{R}} \frac{1}{|(1/4+t^2)(2+t^2)|^{1/2}} dt.$$

Proof. Triangle inequality and pointwise estimate. □

Lemma 3.2.13 (isIntegrable)**.** Let $x > 0$ and $\sigma \in \mathbb{R}$. Then

$$\int_{\mathbb{R}} \frac{x^{\sigma+it}}{(\sigma+it)(1+\sigma+it)} dt$$

is integrable.

Proof. □

Lemma 3.2.14 (tendsto-zero-Lower)**.** Let $x > 0$ and $\sigma', \sigma'' \in \mathbb{R}$. Then

$$\int_{\sigma'}^{\sigma''} \frac{x^{\sigma+it}}{(\sigma+it)(1+\sigma+it)} d\sigma$$

goes to 0 as $t \to -\infty$.

Proof. The numerator is bounded and the denominator tends to infinity. □

Lemma 3.2.15 (tendsto-zero-Upper)**.** Let $x > 0$ and $\sigma', \sigma'' \in \mathbb{R}$. Then

$$\int_{\sigma'}^{\sigma''} \frac{x^{\sigma+it}}{(\sigma+it)(1+\sigma+it)} d\sigma$$

goes to 0 as $t \to \infty$.

Proof. The numerator is bounded and the denominator tends to infinity. □

We are ready for the first case of the Perron formula, namely when $x < 1$:

Lemma 3.2.16 (formulaLtOne)**.** For $x > 0$, $\sigma > 0$, and $x < 1$, we have

$$\frac{1}{2\pi i} \int_{(\sigma)} \frac{x^s}{s(s+1)} ds = 0.$$

Proof. □

The second case is when $x > 1$. Here are some auxiliary lemmata for the second case. TODO: Move to more general section

Lemma 3.2.17 (keyIdentity)**.** Let $x \in \mathbb{R}$ and $s \neq 0, -1$. Then

$$\frac{x^\sigma}{s(1+s)} = \frac{x^\sigma}{s} - \frac{x^\sigma}{1+s}$$

Proof. By ring. □

Lemma 3.2.18 (diffBddAtZero)**.** Let $x > 0$. Then for $0 < c < 1/2$, we have that the function

$$s \mapsto \frac{x^s}{s(s+1)} - \frac{1}{s}$$

is bounded above on the rectangle with corners at $-c - i * c$ and $c + i * c$ (except at $s = 0$).

Proof. Applying Lemma 3.2.17, the function $s \mapsto x^s/s(s+1) - 1/s = x^s/s - x^0/s - x^s/(1+s)$. The last term is bounded for s away from -1. The first two terms are the difference quotient of the function $s \mapsto x^s$ at 0; since it's differentiable, the difference remains bounded as $s \to 0$. □

Lemma 3.2.19 (diffBddAtNegOne)**.** Let $x > 0$. Then for $0 < c < 1/2$, we have that the function

$$s \mapsto \frac{x^s}{s(s+1)} - \frac{-x^{-1}}{s+1}$$

is bounded above on the rectangle with corners at $-1 - c - i * c$ and $-1 + c + i * c$ (except at $s = -1$).

Proof. Applying Lemma 3.2.17, the function $s \mapsto x^s/s(s+1) - x^{-1}/(s+1) = x^s/s - x^s/(s+1) - (-x^{-1})/(s+1)$. The first term is bounded for s away from 0. The last two terms are the difference quotient of the function $s \mapsto x^s$ at -1; since it's differentiable, the difference remains bounded as $s \to -1$. □

Lemma 3.2.20 (residueAtZero)**.** Let $x > 0$. Then for all sufficiently small $c > 0$, we have that

$$\frac{1}{2\pi i} \int_{-c-i*c}^{c+i*c} \frac{x^s}{s(s+1)} ds = 1.$$

Proof. □

Lemma 3.2.21 (residueAtNegOne)**.** Let $x > 0$. Then for all sufficiently small $c > 0$, we have that

$$\frac{1}{2\pi i} \int_{-c-i*c-1}^{c+i*c-1} \frac{x^s}{s(s+1)} ds = -\frac{1}{x}.$$

Proof. Compute the integral. □

Lemma 3.2.22 (residuePull1)**.** For $x > 1$ (of course $x > 0$ would suffice) and $\sigma > 0$, we have

$$\frac{1}{2\pi i} \int_{(\sigma)} \frac{x^s}{s(s+1)} ds = 1 + \frac{1}{2\pi i} \int_{(-1/2)} \frac{x^s}{s(s+1)} ds.$$

Proof. We pull to a square with corners at $-c-i*c$ and $c+i*c$ for $c>0$ sufficiently small. By Lemma 3.2.20, the integral over this square is equal to 1. ☐

Lemma 3.2.23 (residuePull2)**.** For $x>1$, we have

$$\frac{1}{2\pi i}\int_{(-1/2)}\frac{x^s}{s(s+1)}ds = -1/x + \frac{1}{2\pi i}\int_{(-3/2)}\frac{x^s}{s(s+1)}ds.$$

Proof. Pull contour from $(-1/2)$ to $(-3/2)$. ☐

Lemma 3.2.24 (contourPull3)**.** For $x>1$ and $\sigma<-3/2$, we have

$$\frac{1}{2\pi i}\int_{(-3/2)}\frac{x^s}{s(s+1)}ds = \frac{1}{2\pi i}\int_{(\sigma)}\frac{x^s}{s(s+1)}ds.$$

Proof. Pull contour from $(-3/2)$ to (σ). ☐

Lemma 3.2.25 (formulaGtOne)**.** For $x>1$ and $\sigma>0$, we have

$$\frac{1}{2\pi i}\int_{(\sigma)}\frac{x^s}{s(s+1)}ds = 1-1/x.$$

Proof. ☐

The two together give the Perron formula. (Which doesn't need to be a separate lemma.) For $x>0$ and $\sigma>0$, we have

$$\frac{1}{2\pi i}\int_{(\sigma)}\frac{x^s}{s(s+1)}ds = \begin{cases} 1-\frac{1}{x} & \text{if } x>1 \\ 0 & \text{if } x<1 \end{cases}.$$

3.3 Mellin transforms

Lemma 3.3.1 (PartialIntegration)**.** Let f,g be once differentiable functions from $\mathbb{R}_{>0}$ to $\mathbb{C}$ so that fg' and $f'g$ are both integrable, and $f\cdot g(x)\to 0$ as $x\to 0^+,\infty$. Then

$$\int_0^\infty f(x)g'(x)dx = -\int_0^\infty f'(x)g(x)dx.$$

Proof. Partial integration. ☐

In this section, we define the Mellin transform (already in Mathlib, thanks to David Loeffler), prove its inversion formula, and derive a number of important properties of some special functions and bumpfunctions.

Def: (Already in Mathlib) Let f be a function from $\mathbb{R}_{>0}$ to $\mathbb{C}$. We define the Mellin transform of f to be the function $\mathcal{M}(f)$ from $\mathbb{C}$ to $\mathbb{C}$ defined by

$$\mathcal{M}(f)(s) = \int_0^\infty f(x)x^{s-1}dx.$$

[Note: My preferred way to think about this is that we are integrating over the multiplicative group $\mathbb{R}_{>0}$, multiplying by a (not necessarily unitary!) character $|\cdot|^s$, and integrating with respect to the invariant Haar measure dx/x. This is very useful in the kinds of calculations carried out below. But may be more difficult to formalize as things now stand. So we might have clunkier calculations, which "magically" turn out just right - of course they're explained by the aforementioned structure...]

Finally, we need Mellin Convolutions and properties thereof.

Definition 3.3.1 (MellinConvolution)**.** Let f and g be functions from $\mathbb{R}_{>0}$ to $\mathbb{C}$. Then we define the Mellin convolution of f and g to be the function $f * g$ from $\mathbb{R}_{>0}$ to $\mathbb{C}$ defined by

$$(f * g)(x) = \int_0^\infty f(y)g(x/y)\frac{dy}{y}.$$

Let us start with a simple property of the Mellin convolution.

Lemma 3.3.2 (MellinConvolutionSymmetric)**.** Let f and g be functions from $\mathbb{R}_{>0}$ to $\mathbb{R}$ or $\mathbb{C}$, for $x \neq 0$,

$$(f * g)(x) = (g * f)(x).$$

Proof. By Definition 3.3.1,

$$(f * g)(x) = \int_0^\infty f(y)g(x/y)\frac{dy}{y}$$

in which we change variables to $z = x/y$:

$$(f * g)(x) = \int_0^\infty f(x/z)g(z)\frac{dz}{z} = (g * f)(x).$$

□

The Mellin transform of a convolution is the product of the Mellin transforms.

Theorem 3.3.1 (MellinConvolutionTransform)**.** Let f and g be functions from $\mathbb{R}_{>0}$ to $\mathbb{C}$ such that

$$(x, y) \mapsto f(y)\frac{g(x/y)}{y}x^{s-1} \tag{3.1}$$

is absolutely integrable on $[0, \infty)^2$. Then

$$\mathcal{M}(f * g)(s) = \mathcal{M}(f)(s)\mathcal{M}(g)(s).$$

Proof. By Definitions **??** and 3.3.1

$$\mathcal{M}(f * g)(s) = \int_0^\infty \int_0^\infty f(y)g(x/y)x^{s-1}\frac{dy}{y}dx$$

By (3.1) and Fubini's theorem,

$$\mathcal{M}(f * g)(s) = \int_0^\infty \int_0^\infty f(y)g(x/y)x^{s-1}dx\frac{dy}{y}$$

in which we change variables from x to $z = x/y$:

$$\mathcal{M}(f * g)(s) = \int_0^\infty \int_0^\infty f(y)g(z)y^{s-1}z^{s-1}dzdy$$

which, by Definition **??**, is

$$\mathcal{M}(f * g)(s) = \mathcal{M}(f)(s)\mathcal{M}(g)(s).$$

□

The ν function has Mellin transform $\mathcal{M}(\nu)(s)$ which is entire and decays (at least) like $1/|s|$.

[Of course it decays faster than any power of $|s|$, but it turns out that we will just need one power.]

Theorem 3.3.2 (MellinOfPsi)**.** The Mellin transform of ν is

$$\mathcal{M}(\nu)(s) = O\left(\frac{1}{|s|}\right),$$

as $|s| \to \infty$ with $\sigma_1 \leq \Re(s) \leq 2$.

Proof. Integrate by parts:

$$\left|\int_0^\infty \nu(x)x^s \frac{dx}{x}\right| = \left|-\int_0^\infty \nu'(x)\frac{x^s}{s}dx\right|$$

$$\leq \frac{1}{|s|}\int_{1/2}^2 |\nu'(x)|x^{\Re(s)}dx.$$

Since $\Re(s)$ is bounded, the right-hand side is bounded by a constant times $1/|s|$. □

We can make a delta spike out of this bumpfunction, as follows.

Definition 3.3.2 (DeltaSpike)**.** Let ν be a bumpfunction supported in $[1/2, 2]$. Then for any $\epsilon > 0$, we define the delta spike ν_ϵ to be the function from $\mathbb{R}_{>0}$ to $\mathbb{C}$ defined by

$$\nu_\epsilon(x) = \frac{1}{\epsilon}\nu\left(x^{\frac{1}{\epsilon}}\right).$$

This spike still has mass one:

Lemma 3.3.3 (DeltaSpikeMass)**.** For any $\epsilon > 0$, we have

$$\int_0^\infty \nu_\epsilon(x)\frac{dx}{x} = 1.$$

Proof. Substitute $y = x^{1/\epsilon}$, and use the fact that ν has mass one, and that dx/x is Haar measure. □

The Mellin transform of the delta spike is easy to compute.

Theorem 3.3.3 (MellinOfDeltaSpike)**.** For any $\epsilon > 0$, the Mellin transform of ν_ϵ is

$$\mathcal{M}(\nu_\epsilon)(s) = \mathcal{M}(\nu)(\epsilon s).$$

Proof. Substitute $y = x^{1/\epsilon}$, use Haar measure; direct calculation. □

In particular, for $s = 1$, we have that the Mellin transform of ν_ϵ is $1 + O(\epsilon)$.

Corollary 3.3.1 (MellinOfDeltaSpikeAt1)**.** For any $\epsilon > 0$, we have

$$\mathcal{M}(\nu_\epsilon)(1) = \mathcal{M}(\nu)(\epsilon).$$

Proof. This is immediate from the above theorem. □

Lemma 3.3.4 (MellinOfDeltaSpikeAt1-asymp)**.** As $\epsilon \to 0$, we have

$$\mathcal{M}(\nu_\epsilon)(1) = 1 + O(\epsilon).$$

Proof. By Lemma 3.3.1,

$$\mathcal{M}(\nu_\epsilon)(1) = \mathcal{M}(\nu)(\epsilon)$$

which by Definition **??** is

$$\mathcal{M}(\nu)(\epsilon) = \int_0^\infty \nu(x)x^{\epsilon-1}dx.$$

Since $\nu(x)x^{\epsilon-1}$ is integrable (because ν is continuous and compactly supported),

$$\mathcal{M}(\nu)(\epsilon) - \int_0^\infty \nu(x)\frac{dx}{x} = \int_0^\infty \nu(x)(x^{\epsilon-1} - x^{-1})dx.$$

By Taylor's theorem,

$$x^{\epsilon-1} - x^{-1} = O(\epsilon)$$

so, since ν is absolutely integrable,

$$\mathcal{M}(\nu)(\epsilon) - \int_0^\infty \nu(x)\frac{dx}{x} = O(\epsilon).$$

We conclude the proof using Theorem 3.3.5. □

Let $1_{(0,1]}$ be the function from $\mathbb{R}_{>0}$ to $\mathbb{C}$ defined by

$$1_{(0,1]}(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}.$$

This has Mellin transform: [Note: this already exists in mathlib]

Theorem 3.3.4 (MellinOf1)**.** The Mellin transform of $1_{(0,1]}$ is

$$\mathcal{M}(1_{(0,1]})(s) = \frac{1}{s}.$$

Proof. This is a straightforward calculation. □

What will be essential for us is properties of the smooth version of $1_{(0,1]}$, obtained as the Mellin convolution of $1_{(0,1]}$ with ν_ϵ.

Definition 3.3.3 (Smooth1)**.** Let $\epsilon > 0$. Then we define the smooth function $\widetilde{1_\epsilon}$ from $\mathbb{R}_{>0}$ to $\mathbb{C}$ by

$$\widetilde{1_\epsilon} = 1_{(0,1]} * \nu_\epsilon.$$

Proof. Let $c := 2^\epsilon > 1$, in terms of which we wish to prove

$$-1 < c \log c - c.$$

Letting $f(x) := x \log x - x$, we can rewrite this as $f(1) < f(c)$. Since

$$\frac{d}{dx} f(x) = \log x > 0,$$

f is monotone increasing on $[1, \infty)$, and we are done. $\square$

In particular, we have the following two properties.

Lemma 3.3.5 (Smooth1Properties-below)**.** Fix $\epsilon > 0$. There is an absolute constant $c > 0$ so that: If $0 < x \le (1 - c\epsilon)$, then

$$\widetilde{1_\epsilon}(x) = 1.$$

Proof. Opening the definition, we have that the Mellin convolution of $1_{(0,1]}$ with ν_ϵ is

$$\int_0^\infty 1_{(0,1]}(y)\nu_\epsilon(x/y)\frac{dy}{y} = \int_0^1 \nu_\epsilon(x/y)\frac{dy}{y}.$$

The support of ν_ϵ is contained in $[1/2^\epsilon, 2^\epsilon]$, so it suffices to consider $y \in [1/2^\epsilon x, 2^\epsilon x]$ for nonzero contributions. If $x < 2^{-\epsilon}$, then the integral is the same as that over $(0, \infty)$:

$$\int_0^1 \nu_\epsilon(x/y)\frac{dy}{y} = \int_0^\infty \nu_\epsilon(x/y)\frac{dy}{y},$$

in which we change variables to $z = x/y$ (using $x > 0$):

$$\int_0^\infty \nu_\epsilon(x/y)\frac{dy}{y} = \int_0^\infty \nu_\epsilon(z)\frac{dz}{z},$$

which is equal to one by Lemma 3.3.3. We then choose

$$c := \log 2,$$

which satisfies

$$c > \frac{1 - 2^{-\epsilon}}{\epsilon}$$

by Lemma **??**, so

$$1 - c\epsilon < 2^{-\epsilon}.$$

$\square$

Lemma 3.3.6 (Smooth1Properties-above)**.** Fix $0 < \epsilon < 1$. There is an absolute constant $c > 0$ so that: if $x \ge (1 + c\epsilon)$, then

$$\widetilde{1_\epsilon}(x) = 0.$$

Proof. Again the Mellin convolution is

$$\int_0^1 \nu_\epsilon(x/y)\frac{dy}{y},$$

but now if $x > 2^\epsilon$, then the support of ν_ϵ is disjoint from the region of integration, and hence the integral is zero. We choose

$$c := 2\log 2.$$

By Lemma **??**,

$$c > 2\frac{1-2^{-\epsilon}}{\epsilon} > 2^\epsilon\frac{1-2^{-\epsilon}}{\epsilon} = \frac{2^\epsilon - 1}{\epsilon},$$

so

$$1 + c\epsilon > 2^\epsilon.$$

□

Lemma 3.3.7 (Smooth1Nonneg)**.** If ν is nonnegative, then $\widetilde{1_\epsilon}(x)$ is nonnegative.

Proof. By Definitions 3.3.3, 3.3.1 and 3.3.2

$$\widetilde{1_\epsilon}(x) = \int_0^\infty 1_{(0,1]}(y)\frac{1}{\epsilon}\nu((x/y)^{\frac{1}{\epsilon}})\frac{dy}{y}$$

and all the factors in the integrand are nonnegative. □

Lemma 3.3.8 (Smooth1LeOne)**.** If ν is nonnegative and has mass one, then $\widetilde{1_\epsilon}(x) \leq 1$, $\forall x > 0$.

Proof. By Definitions 3.3.3, 3.3.1 and 3.3.2

$$\widetilde{1_\epsilon}(x) = \int_0^\infty 1_{(0,1]}(y)\frac{1}{\epsilon}\nu((x/y)^{\frac{1}{\epsilon}})\frac{dy}{y}$$

and since $1_{(0,1]}(y) \leq 1$, and all the factors in the integrand are nonnegative,

$$\widetilde{1_\epsilon}(x) \leq \int_0^\infty \frac{1}{\epsilon}\nu((x/y)^{\frac{1}{\epsilon}})\frac{dy}{y}$$

(because in mathlib the integral of a non-integrable function is 0, for the inequality above to be true, we must prove that $\nu((x/y)^{\frac{1}{\epsilon}})/y$ is integrable; this follows from the computation below). We then change variables to $z = (x/y)^{\frac{1}{\epsilon}}$:

$$\widetilde{1_\epsilon}(x) \leq \int_0^\infty \nu(z)\frac{dz}{z}$$

which by Theorem 3.3.5 is 1. □

Combining the above, we have the following three Main Lemmata of this section on the Mellin transform of $\widetilde{1_\epsilon}$.

Lemma 3.3.9 (MellinOfSmooth1a)**.** Fix $\epsilon > 0$. Then the Mellin transform of $\widetilde{1_\epsilon}$ is

$$\mathcal{M}(\widetilde{1_\epsilon})(s) = \frac{1}{s}\left(\mathcal{M}(\nu)\left(\epsilon s\right)\right).$$

Proof. By Definition 3.3.3,

$$\mathcal{M}(\widetilde{1_\epsilon})(s) = \mathcal{M}(1_{(0,1]} * \nu_\epsilon)(s).$$

We wish to apply Theorem 3.3.1. To do so, we must prove that

$$(x, y) \mapsto 1_{(0,1]}(y)\nu_\epsilon(x/y)/y$$

is integrable on $[0, \infty)^2$. It is actually easier to do this for the convolution: $\nu_\epsilon * 1_{(0,1]}$, so we use Lemma 3.3.2: for $x \neq 0$,

$$1_{(0,1]} * \nu_\epsilon(x) = \nu_\epsilon * 1_{(0,1]}(x).$$

Now, for $x = 0$, both sides of the equation are 0, so the equation also holds for $x = 0$. Therefore,

$$\mathcal{M}(\widetilde{1_\epsilon})(s) = \mathcal{M}(\nu_\epsilon * 1_{(0,1]})(s).$$

Now,

$$(x, y) \mapsto \nu_\epsilon(y)1_{(0,1]}(x/y)\frac{x^{s-1}}{y}$$

has compact support that is bounded away from $y = 0$ (specifically $y \in [2^{-\epsilon}, 2^\epsilon]$ and $x \in (0, y]$), so it is integrable. We can thus apply Theorem 3.3.1 and find

$$\mathcal{M}(\widetilde{1_\epsilon})(s) = \mathcal{M}(\nu_\epsilon)(s)\mathcal{M}(1_{(0,1]})(s).$$

By Lemmas 3.3.4 and 3.3.3,

$$\mathcal{M}(\widetilde{1_\epsilon})(s) = \frac{1}{s}\mathcal{M}(\nu)(\epsilon s).$$

□

Lemma 3.3.10 (MellinOfSmooth1b)**.** Given $0 < \sigma_1 \leq \sigma_2$, for any s such that $\sigma_1 \leq \mathcal{R}e(s) \leq \sigma_2$, we have

$$\mathcal{M}(\widetilde{1_\epsilon})(s) = O\left(\frac{1}{\epsilon|s|^2}\right).$$

Proof. Use Lemma 3.3.9 and the bound in Lemma 3.3.2. □

Lemma 3.3.11 (MellinOfSmooth1c)**.** At $s = 1$, we have

$$\mathcal{M}(\widetilde{1_\epsilon})(1) = 1 + O(\epsilon)).$$

Proof. Follows from Lemmas 3.3.9, 3.3.1 and 3.3.4. □

Lemma 3.3.12 (Smooth1ContinuousAt)**.** Fix a nonnegative, continuously differentiable function F on $\mathbb{R}$ with support in $[1/2, 2]$. Then for any $\epsilon > 0$, the function $x \mapsto \int_{(0,\infty)} x^{1+it}\widetilde{1_\epsilon}(x)dx$ is continuous at any $y > 0$.

Proof. Use Lemma **??** to write $\widetilde{1_\epsilon}(x)$ as an integral over an integral near 1, in particular avoiding the singularity at 0. The integrand may be bounded by $2^\epsilon\nu_\epsilon(t)$ which is independent of x and we can use dominated convergence to prove continuity. □

Let ν be a bumpfunction.

Theorem 3.3.5 (SmoothExistence)**.** There exists a smooth (once differentiable would be enough), nonnegative "bumpfunction" ν, supported in $[1/2, 2]$ with total mass one:

$$\int_0^\infty \nu(x)\frac{dx}{x} = 1.$$

Proof. Same idea as Urysohn-type argument. □

3.4 Zeta Bounds

Already on Mathlib (with a shortened proof):

Theorem 3.4.1 (hasDerivAt-conj-conj)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be a complex differentiable function at $p \in \mathbb{C}$ with derivative a. Then the function $g(z) = \overline{f(\overline{z})}$ is complex differentiable at $\overline{p}$ with derivative $\overline{a}$.

Proof. We expand the definition of the derivative and compute. □

Submitted to Mathlib:

Theorem 3.4.2 (deriv-conj-conj)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be a function at $p \in \mathbb{C}$ with derivative a. Then the derivative of the function $g(z) = \overline{f(\overline{z})}$ at $\overline{p}$ is $\overline{a}$.

Proof. We proceed by case analysis on whether f is differentiable at p. If f is differentiable at p, then we can apply the previous theorem. If f is not differentiable at p, then neither is g, and both derivatives have the default value of zero. □

Theorem 3.4.3 (conj-riemannZeta-conj-aux1)**.** Conjugation symmetry of the Riemann zeta function in the half-plane of convergence. Let $s \in \mathbb{C}$ with $\Re(s) > 1$. Then $\overline{\zeta(\overline{s})} = \zeta(s)$.

Proof. We expand the definition of the Riemann zeta function as a series and find that the two sides are equal term by term. □

Theorem 3.4.4 (conj-riemannZeta-conj)**.** Conjugation symmetry of the Riemann zeta function. Let $s \in \mathbb{C}$. Then

$$\overline{\zeta(\overline{s})} = \zeta(s).$$

Proof. By the previous lemma, the two sides are equal on the half-plane $\{s \in \mathbb{C} : \Re(s) > 1\}$. Then, by analytic continuation, they are equal on the whole complex plane. □

Theorem 3.4.5 (riemannZeta-conj)**.** Conjugation symmetry of the Riemann zeta function. Let $s \in \mathbb{C}$. Then

$$\zeta(\overline{s}) = \overline{\zeta(s)}.$$

Proof. This follows as an immediate corollary of Theorem 3.4.4. □

Theorem 3.4.6 (deriv-riemannZeta-conj)**.** Conjugation symmetry of the derivative of the Riemann zeta function. Let $s \in \mathbb{C}$. Then

$$\zeta'(\overline{s}) = \overline{\zeta'(s)}.$$

Proof. We apply the derivative conjugation symmetry to the Riemann zeta function and use the conjugation symmetry of the Riemann zeta function itself. □

Theorem 3.4.7 (intervalIntegral-conj)**.** The conjugation symmetry of the interval integral. Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable function, and let $a, b \in \mathbb{R}$. Then

$$\int_a^b \overline{f(x)}\, dx = \overline{\int_a^b f(x)\, dx}.$$

Proof. We unfold the interval integral into an integral over a uIoc and use the conjugation property of integrals. □

We record here some prelimiaries about the zeta function and general holomorphic functions.

Theorem 3.4.8 (ResidueOfTendsTo)**.** If a function f is holomorphic in a neighborhood of p and $\lim_{s\to p}(s-p)f(s) = A$, then $f(s) = \frac{A}{s-p} + O(1)$ near p.

Proof. The function $(s-p)\cdot f(s)$ bounded, so by Theorem 3.1.1, there is a holomorphic function, g, say, so that $(s-p)f(s) = g(s)$ in a neighborhood of $s = p$, and $g(p) = A$. Now because g is holomorphic, near $s = p$, we have $g(s) = A + O(s-p)$. Then when you divide by $(s-p)$, you get $f(s) = A/(s-p) + O(1)$. ☐

Theorem 3.4.9 (riemannZetaResidue)**.** The Riemann zeta function $\zeta(s)$ has a simple pole at $s = 1$ with residue 1. In particular, the function $\zeta(s) - \frac{1}{s-1}$ is bounded in a neighborhood of $s = 1$.

Proof. From `riemannZeta_residue_one` (in Mathlib), we know that $(s-1)\zeta(s)$ goes to 1 as $s \to 1$. Now apply Theorem 3.4.8. (This can also be done using ζ_0 below, which is expressed as $1/(s-1)$ plus things that are holomorphic for $\Re(s) > 0$...) ☐

Theorem 3.4.10 (nonZeroOfBddAbove)**.** If a function f has a simple pole at a point p with residue $A \neq 0$, then f is nonzero in a punctured neighborhood of p.

Proof. We know that $f(s) = \frac{A}{s-p} + O(1)$ near p, so we can write

$$f(s) = \left(f(s) - \frac{A}{s-p}\right) + \frac{A}{s-p}.$$

The first term is bounded, say by M, and the second term goes to ∞ as $s \to p$. Therefore, there exists a neighborhood V of p such that for all $s \in V \setminus \{p\}$, we have $f(s) \neq 0$. ☐

Theorem 3.4.11 (logDerivResidue)**.** If f is holomorphic in a neighborhood of p, and there is a simple pole at p, then f'/f has a simple pole at p with residue -1:

$$\frac{f'(s)}{f(s)} = \frac{-1}{s-p} + O(1).$$

Proof. Using Theorem 3.1.1, there is a function g holomorphic near p, for which $f(s) = A/(s-p) + g(s) = h(s)/(s-p)$. Here $h(s) := A + g(s)(s-p)$ which is nonzero in a neighborhood of p (since h goes to A which is nonzero). Then $f'(s) = (h'(s)(s-p) - h(s))/(s-p)^2$, and we can compute the quotient:

$$\frac{f'(s)}{f(s)} + 1/(s-p) = \frac{h'(s)(s-p) - h(s)}{h(s)} \cdot \frac{1}{(s-p)} + 1/(s-p) = \frac{h'(s)}{h(s)}.$$

Since h is nonvanishing near p, this remains bounded in a neighborhood of p. ☐

Theorem 3.4.12 (BddAbove-to-IsBigO)**.** If f is bounded above in a punctured neighborhood of p, then f is $O(1)$ in that neighborhood.

Proof. Elementary. ☐

Let's also record that if a function f has a simple pole at p with residue A, and g is holomorphic near p, then the residue of $f \cdot g$ is $A \cdot g(p)$.

Theorem 3.4.13 (ResidueMult)**.** If f has a simple pole at p with residue A, and g is holomorphic near p, then the residue of $f \cdot g$ at p is $A \cdot g(p)$. That is, we assume that

$$f(s) = \frac{A}{s-p} + O(1)$$

near p, and that g is holomorphic near p. Then

$$f(s) \cdot g(s) = \frac{A \cdot g(p)}{s-p} + O(1).$$

Proof. Elementary calculation.

$$f(s) * g(s) - \frac{A * g(p)}{s-p} = \left(f(s) * g(s) - \frac{A * g(s)}{s-p}\right) + \left(\frac{A * g(s) - A * g(p)}{s-p}\right).$$

The first term is $g(s)(f(s) - \frac{A}{s-p})$, which is bounded near p by the assumption on f and the fact that g is holomorphic near p. The second term is A times the log derivative of g at p, which is bounded by the assumption that g is holomorphic. □

As a corollary, the log derivative of the Riemann zeta function has a simple pole at $s = 1$:

Theorem 3.4.14 (riemannZetaLogDerivResidue)**.** The log derivative of the Riemann zeta function $\zeta(s)$ has a simple pole at $s = 1$ with residue -1: $-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} = O(1)$.

Proof. This follows from Theorem 3.4.11 and Theorem 3.4.9. □

Definition 3.4.1 (riemannZeta0)**.** For any natural $N \geq 1$, we define

$$\zeta_0(N, s) := \sum_{1 \leq n \leq N} \frac{1}{n^s} + \frac{-N^{1-s}}{1-s} + \frac{-N^{-s}}{2} + s \int_N^\infty \frac{\lfloor x \rfloor + 1/2 - x}{x^{s+1}}\, dx$$

Lemma 3.4.1 (sum-eq-int-deriv)**.** Let $a < b$, and let ϕ be continuously differentiable on $[a, b]$. Then

$$\sum_{a<n\leq b} \phi(n) = \int_a^b \phi(x)\, dx + \left(\lfloor b \rfloor + \frac{1}{2} - b\right) \phi(b) - \left(\lfloor a \rfloor + \frac{1}{2} - a\right) \phi(a) - \int_a^b \left(\lfloor x \rfloor + \frac{1}{2} - x\right) \phi'(x)\, dx.$$

Proof. Specialize Abel summation from Mathlib to the trivial arithmetic function and then manipulate integrals. □

Lemma 3.4.2 (ZetaSum-aux1)**.** Let $0 < a < b$ be natural numbers and $s \in \mathbb{C}$ with $s \neq 1$ and $s \neq 0$. Then

$$\sum_{a<n\leq b} \frac{1}{n^s} = \frac{b^{1-s} - a^{1-s}}{1-s} + \frac{b^{-s} - a^{-s}}{2} + s \int_a^b \frac{\lfloor x \rfloor + 1/2 - x}{x^{s+1}}\, dx.$$

Proof. Apply Lemma 3.4.1 to the function $x \mapsto x^{-s}$. □

Lemma 3.4.3 (ZetaBnd-aux1a)**.** For any $0 < a < b$ and $s \in \mathbb{C}$ with $\sigma = \Re(s) > 0$,

$$\int_a^b \left| \frac{\lfloor x \rfloor + 1/2 - x}{x^{s+1}}\, dx \right| \leq \frac{a^{-\sigma} - b^{-\sigma}}{\sigma}.$$

Proof. Apply the triangle inequality

$$\left|\int_a^b \frac{\lfloor x\rfloor + 1/2 - x}{x^{s+1}}\, dx\right| \le \int_a^b \frac{1}{x^{\sigma+1}}\, dx,$$

and evaluate the integral. ☐

Lemma 3.4.4 (ZetaSum-aux2)**.** Let N be a natural number and $s \in \mathbb{C}$, $\Re(s) > 1$. Then

$$\sum_{N<n} \frac{1}{n^s} = \frac{-N^{1-s}}{1-s} + \frac{-N^{-s}}{2} + s\int_N^\infty \frac{\lfloor x\rfloor + 1/2 - x}{x^{s+1}}\, dx.$$

Proof. Apply Lemma 3.4.2 with $a = N$ and $b \to \infty$. ☐

Lemma 3.4.5 (ZetaBnd-aux1b)**.** For any $N \ge 1$ and $s = \sigma + tI \in \mathbb{C}$, $\sigma > 0$,

$$\left|\int_N^\infty \frac{\lfloor x\rfloor + 1/2 - x}{x^{s+1}}\, dx\right| \le \frac{N^{-\sigma}}{\sigma}.$$

Proof. Apply Lemma 3.4.3 with $a = N$ and $b \to \infty$. ☐

Lemma 3.4.6 (ZetaBnd-aux1)**.** For any $N \ge 1$ and $s = \sigma + tI \in \mathbb{C}$, $\sigma =\in (0, 2], 2 < |t|$,

$$\left|s\int_N^\infty \frac{\lfloor x\rfloor + 1/2 - x}{x^{s+1}}\, dx\right| \le 2|t|\frac{N^{-\sigma}}{\sigma}.$$

Proof. Apply Lemma 3.4.5 and estimate $|s| \ll |t|$. ☐

Big-Oh version of Lemma 3.4.6.

Lemma 3.4.7 (ZetaBnd-aux1p)**.** For any $N \ge 1$ and $s = \sigma + tI \in \mathbb{C}$, $\sigma =\in (0, 2], 2 < |t|$,

$$\left|s\int_N^\infty \frac{\lfloor x\rfloor + 1/2 - x}{x^{s+1}}\, dx\right| \ll |t|\frac{N^{-\sigma}}{\sigma}.$$

Proof. Apply Lemma 3.4.5 and estimate $|s| \ll |t|$. ☐

Theorem 3.4.15 (HolomorphicOn-riemannZeta0)**.** For any $N \ge 1$, the function $\zeta_0(N, s)$ is holomorphic on $\{s \in \mathbb{C} \mid \Re(s) > 0 \wedge s \ne 1\}$.

Proof. The function $\zeta_0(N, s)$ is a finite sum of entire functions, plus an integral that's absolutely convergent on $\{s \in \mathbb{C} \mid \Re(s) > 0 \wedge s \ne 1\}$ by Lemma 3.4.5. ☐

Lemma 3.4.8 (isPathConnected-aux)**.** The set $\{s \in \mathbb{C} \mid \Re(s) > 0 \wedge s \ne 1\}$ is path-connected.

Proof. Construct explicit paths from 2 to any point, either a line segment or two joined ones. ☐

Lemma 3.4.9 (Zeta0EqZeta)**.** For $\Re(s) > 0$, $s \ne 1$, and for any N,

$$\zeta_0(N, s) = \zeta(s).$$

Proof. Use Lemma 3.4.4 and the Definition 3.4.1. ☐

Lemma 3.4.10 (ZetaBnd-aux2)**.** Given $n \le t$ and σ with $1 - A/\log t \le \sigma$, we have that

$$|n^{-s}| \le n^{-1}e^{A}.$$

Proof. Use $|n^{-s}| = n^{-\sigma} = e^{-\sigma \log n} \le \exp(-\left(1 - \frac{A}{\log t}\right)\log n) \le n^{-1}e^{A}$, since $n \le t$. □

Lemma 3.4.11 (ZetaUpperBnd)**.** For any $s = \sigma + tI \in \mathbb{C}$, $1/2 \le \sigma \le 2, 3 < |t|$ and any $0 < A < 1$ sufficiently small, and $1 - A/\log|t| \le \sigma$, we have

$$|\zeta(s)| \ll \log t.$$

Proof. First replace $\zeta(s)$ by $\zeta_0(N, s)$ for $N = \lfloor |t| \rfloor$. We estimate:

$$|\zeta_0(N,s)| \ll \sum_{1\le n\le|t|} |n^{-s}| + \frac{-|t|^{1-\sigma}}{|1-s|} + \frac{-|t|^{-\sigma}}{2} + |t|\cdot|t|^{-\sigma}/\sigma$$
$$\ll e^{A}\sum_{1\le n<|t|} n^{-1} + |t|^{1-\sigma}$$

, where we used Lemma 3.4.10 and Lemma 3.4.6. The first term is $\ll \log|t|$. For the second term, estimate

$$|t|^{1-\sigma} \le |t|^{1-(1-A/\log|t|)} = |t|^{A/\log|t|} \ll 1.$$

□

Lemma 3.4.12 (DerivUpperBnd-aux7)**.** For any $s = \sigma + tI \in \mathbb{C}$, $1/2 \le \sigma \le 2, 3 < |t|$, and any $0 < A < 1$ sufficiently small, and $1 - A/\log|t| \le \sigma$, we have

$$\left\| s \cdot \int_N^\infty \left(\lfloor x \rfloor + \frac{1}{2} - x\right) \cdot x^{-s-1} \cdot (-\log x) \right\| \le 2\cdot|t|\cdot N^{-\sigma}/\sigma \cdot \log|t|.$$

Proof. Estimate $|s| = |\sigma + tI|$ by $|s| \le 2 + |t| \le 2|t|$ (since $|t| > 3$). Estimating $|\lfloor x \rfloor + 1/2 - x|$ by 1, and using $|x^{-s-1}| = x^{-\sigma-1}$, we have

$$\left\| s \cdot \int_N^\infty \left(\lfloor x \rfloor + \frac{1}{2} - x\right) \cdot x^{-s-1} \cdot (-\log x) \right\| \le 2\cdot|t| \int_N^\infty x^{-\sigma}\cdot(\log x).$$

For the last integral, integrate by parts, getting:

$$\int_N^\infty x^{-\sigma-1}\cdot(\log x) = \frac{1}{\sigma}N^{-\sigma}\cdot\log N + \frac{1}{\sigma^2}\cdot N^{-\sigma}.$$

Now use $\log N \le \log|t|$ to get the result. □

Lemma 3.4.13 (ZetaDerivUpperBnd)**.** For any $s = \sigma + tI \in \mathbb{C}$, $1/2 \le \sigma \le 2, 3 < |t|$, there is an $A > 0$ so that for $1 - A/\log t \le \sigma$, we have

$$|\zeta'(s)| \ll \log^2 t.$$

Proof. First replace $\zeta(s)$ by $\zeta_0(N, s)$ for $N = \lfloor |t| \rfloor$. Differentiating term by term, we get:

$$\zeta'(s) = -\sum_{1\le n<N} n^{-s}\log n + \frac{N^{1-s}}{(1-s)^2} + \frac{N^{1-s}\log N}{1-s} + \frac{N^{-s}\log N}{2} + \int_N^\infty \frac{\lfloor x \rfloor + 1/2 - x}{x^{s+1}}\,dx - s\int_N^\infty \log x \frac{\lfloor x \rfloor + 1/2 - x}{x^{s+1}}\,dx$$

Estimate as before, with an extra factor of $\log|t|$. □

Lemma 3.4.14 (ZetaNear1BndFilter)**.** As $\sigma \to 1^+$,

$$|\zeta(\sigma)| \ll 1/(\sigma - 1).$$

Proof. Zeta has a simple pole at $s = 1$. Equivalently, $\zeta(s)(s-1)$ remains bounded near 1. Lots of ways to prove this. Probably the easiest one: use the expression for $\zeta_0(N, s)$ with $N = 1$ (the term $N^{1-s}/(1-s)$ being the only unbounded one). □

Lemma 3.4.15 (ZetaNear1BndExact)**.** There exists a $c > 0$ such that for all $1 < \sigma \leq 2$,

$$|\zeta(\sigma)| \leq c/(\sigma - 1).$$

Proof. Split into two cases, use Lemma 3.4.14 for σ sufficiently small and continuity on a compact interval otherwise. □

Lemma 3.4.16 (ZetaLowerBound3)**.** There exists a $c > 0$ such that for all $1 < \sigma <= 2$ and $3 < |t|$,

$$c\frac{(\sigma-1)^{3/4}}{(\log|t|)^{1/4}} \leq |\zeta(\sigma + tI)|.$$

Proof. Combine Lemma **??** with upper bounds for $|\zeta(\sigma)|$ (from Lemma 3.4.15) and $|\zeta(\sigma + 2it)|$ (from Lemma 3.4.11). □

Lemma 3.4.17 (ZetaInvBound1)**.** For all $\sigma > 1$,

$$1/|\zeta(\sigma + it)| \leq |\zeta(\sigma)|^{3/4}|\zeta(\sigma + 2it)|^{1/4}$$

Proof. The identity

$$1 \leq |\zeta(\sigma)|^3|\zeta(\sigma + it)|^4|\zeta(\sigma + 2it)|$$

for $\sigma > 1$ is already proved by Michael Stoll in the EulerProducts PNT file. □

Lemma 3.4.18 (ZetaInvBound2)**.** For $\sigma > 1$ (and $\sigma \leq 2$),

$$1/|\zeta(\sigma + it)| \ll (\sigma - 1)^{-3/4}(\log|t|)^{1/4},$$

as $|t| \to \infty$.

Proof. Combine Lemma 3.4.17 with the bounds in Lemmata 3.4.15 and 3.4.11. □

Lemma 3.4.19 (Zeta-eq-int-derivZeta)**.** For any $t \neq 0$ (so we don't pass through the pole), and $\sigma_1 < \sigma_2$,

$$\int_{\sigma_1}^{\sigma_2} \zeta'(\sigma + it)dt = \zeta(\sigma_2 + it) - \zeta(\sigma_1 + it).$$

Proof. This is the fundamental theorem of calculus. □

Lemma 3.4.20 (Zeta-diff-Bnd)**.** For any $A > 0$ sufficiently small, there is a constant $C > 0$ so that whenever $1 - A/\log t \leq \sigma_1 < \sigma_2 \leq 2$ and $3 < |t|$, we have that:

$$|\zeta(\sigma_2 + it) - \zeta(\sigma_1 + it)| \leq C(\log|t|)^2(\sigma_2 - \sigma_1).$$

Proof. Use Lemma 3.4.19 and estimate trivially using Lemma 3.4.13. □

Lemma 3.4.21 (ZetaInvBnd)**.** For any $A > 0$ sufficiently small, there is a constant $C > 0$ so that whenever $1 - A/\log^9 |t| \le \sigma < 1 + A/\log^9 |t|$ and $3 < |t|$, we have that:

$$1/|\zeta(\sigma + it)| \le C \log^7 |t|.$$

Proof. Let σ be given in the prescribed range, and set $\sigma' := 1 + A/\log^9 |t|$. Then

$$|\zeta(\sigma+it)| \ge |\zeta(\sigma'+it)| - |\zeta(\sigma+it) - \zeta(\sigma'+it)| \ge C(\sigma'-1)^{3/4} \log |t|^{-1/4} - C\log^2 |t|(\sigma' - \sigma)$$

$$\ge CA^{3/4} \log |t|^{-7} - C \log^2 |t|(2A/\log^9 |t|),$$

where we used Lemma 3.4.18 and Lemma 3.4.20. Now by making A sufficiently small (in particular, something like $A = 1/16$ should work), we can guarantee that

$$|\zeta(\sigma + it)| \ge \frac{C}{2}(\log |t|)^{-7},$$

as desired. □

Annoyingly, it is not immediate from this that ζ doesn't vanish there! That's because $1/0 = 0$ in Lean. So we give a second proof of the same fact (refactor this later), with a lower bound on ζ instead of upper bound on $1/\zeta$.

Lemma 3.4.22 (ZetaLowerBnd)**.** For any $A > 0$ sufficiently small, there is a constant $C > 0$ so that whenever $1 - A/\log^9 |t| \le \sigma < 1$ and $3 < |t|$, we have that:

$$|\zeta(\sigma + it)| \ge C \log^7 |t|.$$

Proof. Follow same argument. □

Now we get a zero free region.

Lemma 3.4.23 (ZetaZeroFree)**.** There is an $A > 0$ so that for $1 - A/\log^9 |t| \le \sigma < 1$ and $3 < |t|$,

$$\zeta(\sigma + it) \ne 0.$$

Proof. Apply Lemma 3.4.22. □

Lemma 3.4.24 (LogDerivZetaBnd)**.** There is an $A > 0$ so that for $1 - A/\log^9 |t| \le \sigma < 1 + A/\log^9 |t|$ and $3 < |t|$,

$$|\frac{\zeta'}{\zeta}(\sigma + it)| \ll \log^9 |t|.$$

Proof. Combine the bound on $|\zeta'|$ from Lemma 3.4.13 with the bound on $1/|\zeta|$ from Lemma 3.4.21. □

Theorem 3.4.16 (ZetaNoZerosOn1Line)**.** The zeta function does not vanish on the 1-line.

Proof. This fact is already proved in Stoll's work. □

Then, since ζ doesn't vanish on the 1-line, there is a $\sigma < 1$ (depending on T), so that the box $[\sigma, 1] \times_{\mathbb{C}} [-T, T]$ is free of zeros of ζ.

Lemma 3.4.25 (ZetaNoZerosInBox)**.** For any $T > 0$, there is a constant $\sigma < 1$ so that

$$\zeta(\sigma' + it) \neq 0$$

for all $|t| \leq T$ and $\sigma' \geq \sigma$.

Proof. Assume not. Then there is a sequence $|t_n| \leq T$ and $\sigma_n \to 1$ so that $\zeta(\sigma_n + it_n) = 0$. By compactness, there is a subsequence $t_{n_k} \to t_0$ along which $\zeta(\sigma_{n_k} + it_{n_k}) = 0$. If $t_0 \neq 0$, use the continuity of ζ to get that $\zeta(1 + it_0) = 0$; this is a contradiction. If $t_0 = 0$, ζ blows up near 1, so can't be zero nearby. □

We now prove that there's an absolute constant σ_0 so that ζ'/ζ is holomorphic on a rectangle $[\sigma_2, 2] \times_{\mathbb{C}} [-3, 3] \setminus \{1\}$.

Lemma 3.4.26 (LogDerivZetaHolcSmallT)**.** There is a $\sigma_2 < 1$ so that the function

$$\frac{\zeta'}{\zeta}(s)$$

is holomorphic on $\{\sigma_2 \leq \Re s \leq 2, |\Im s| \leq 3\} \setminus \{1\}$.

Proof. The derivative of ζ is holomorphic away from $s = 1$; the denominator $\zeta(s)$ is nonzero in this range by Lemma 3.4.25. □

Lemma 3.4.27 (LogDerivZetaHolcLargeT)**.** There is an $A > 0$ so that for all $T > 3$, the function $\frac{\zeta'}{\zeta}(s)$ is holomorphic on $\{1 - A/\log^9 T \leq \Re s \leq 2, |\Im s| \leq T\} \setminus \{1\}$.

Proof. The derivative of ζ is holomorphic away from $s = 1$; the denominator $\zeta(s)$ is nonzero in this range by Lemma 3.4.23. □

Lemma 3.4.28 (LogDerivZetaBndUnif)**.** There exist $A, C > 0$ such that

$$|\frac{\zeta'}{\zeta}(\sigma + it)| \leq C \log |t|^9$$

whenever $|t| > 3$ and $\sigma > 1 - A/\log |t|^9$.

Proof. For σ close to 1 use Lemma 3.4.24, otherwise estimate trivially. □

3.5 Proof of Medium PNT

The approach here is completely standard. We follow the use of $\mathcal{M}(\widetilde{1_\epsilon})$ as in [Kontorovich 2015].

Definition 3.5.1 (ChebyshevPsi)**.** The (second) Chebyshev Psi function is defined as

$$\psi(x) := \sum_{n \leq x} \Lambda(n),$$

where $\Lambda(n)$ is the von Mangoldt function.

It has already been established that zeta doesn't vanish on the 1 line, and has a pole at $s = 1$ of order 1. We also have the following.

Theorem 3.5.1 (LogDerivativeDirichlet)**.** We have that, for $\Re(s) > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Proof. Already in Mathlib. □

The main object of study is the following inverse Mellin-type transform, which will turn out to be a smoothed Chebyshev function.

Definition 3.5.2 (SmoothedChebyshev)**.** Fix $\epsilon > 0$, and a bumpfunction supported in $[1/2, 2]$. Then we define the smoothed Chebyshev function ψ_ϵ from $\mathbb{R}_{>0}$ to $\mathbb{C}$ by

$$\psi_\epsilon(X) = \frac{1}{2\pi i} \int_{(\sigma)} \frac{-\zeta'(s)}{\zeta(s)} \mathcal{M}(\widetilde{1_\epsilon})(s) X^s ds,$$

where we'll take $\sigma = 1 + 1/\log X$.

Lemma 3.5.1 (SmoothedChebyshevDirichlet-aux-integrable)**.** Fix a nonnegative, continuously differentiable function F on $\mathbb{R}$ with support in $[1/2, 2]$, and total mass one, $\int_{(0,\infty)} F(x)/x dx = 1$. Then for any $\epsilon > 0$, and $\sigma \in (1, 2]$, the function

$$x \mapsto \mathcal{M}(\widetilde{1_\epsilon})(\sigma + ix)$$

is integrable on $\mathbb{R}$.

Proof. By Lemma 3.3.10 the integrand is $O(1/t^2)$ as $t \to \infty$ and hence the function is integrable. □

Lemma 3.5.2 (SmoothedChebyshevDirichlet-aux-tsum-integral)**.** Fix a nonnegative, continuously differentiable function F on $\mathbb{R}$ with support in $[1/2, 2]$, and total mass one, $\int_{(0,\infty)} F(x)/x dx = 1$. Then for any $\epsilon > 0$ and $\sigma \in (1, 2]$, the function $x \mapsto \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma+it}} \mathcal{M}(\widetilde{1_\epsilon})(\sigma + it) x^{\sigma+it}$ is equal to $\sum_{n=1}^{\infty} \int_{(0,\infty)} \frac{\Lambda(n)}{n^{\sigma+it}} \mathcal{M}(\widetilde{1_\epsilon})(\sigma + it) x^{\sigma+it}$.

Proof. Interchange of summation and integration. □

Theorem 3.5.2 (SmoothedChebyshevDirichlet)**.** We have that

$$\psi_\epsilon(X) = \sum_{n=1}^{\infty} \Lambda(n) \widetilde{1_\epsilon}(n/X).$$

Proof. We have that

$$\psi_\epsilon(X) = \frac{1}{2\pi i} \int_{(2)} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \mathcal{M}(\widetilde{1_\epsilon})(s) X^s ds.$$

We have enough decay (thanks to quadratic decay of $\mathcal{M}(\widetilde{1_\epsilon})$) to justify the interchange of summation and integration. We then get

$$\psi_\epsilon(X) = \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi i} \int_{(2)} \mathcal{M}(\widetilde{1_\epsilon})(s) (n/X)^{-s} ds$$

and apply the Mellin inversion formula. □

The smoothed Chebyshev function is close to the actual Chebyshev function.

Theorem 3.5.3 (SmoothedChebyshevClose)**.** We have that

$$\psi_\epsilon(X) = \psi(X) + O(\epsilon X \log X).$$

Proof. Take the difference. By Lemma 3.3.6 and 3.3.5, the sums agree except when $1 - c\epsilon \leq n/X \leq 1 + c\epsilon$. This is an interval of length $\ll \epsilon X$, and the summands are bounded by $\Lambda(n) \ll \log X$.

□

Returning to the definition of ψ_ϵ, fix a large T to be chosen later, and set $\sigma_0 = 1 + 1/logX$, $\sigma_1 = 1 - A/\log T^9$, and $\sigma_2 < \sigma_1$ a constant. Pull contours (via rectangles!) to go from $\sigma_0 - i\infty$ up to $\sigma_0 - iT$, then over to $\sigma_1 - iT$, up to $\sigma_1 - 3i$, over to $\sigma_2 - 3i$, up to $\sigma_2 + 3i$, back over to $\sigma_1 + 3i$, up to $\sigma_1 + iT$, over to $\sigma_0 + iT$, and finally up to $\sigma_0 + i\infty$.

In the process, we will pick up the residue at $s = 1$. We will do this in several stages. Here the interval integrals are defined as follows:

Definition 3.5.3 (I)**.**

$$I_1(\nu, \epsilon, X, T) := \frac{1}{2\pi i} \int_{-\infty}^{-T} \left(\frac{-\zeta'}{\zeta}(\sigma_0 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_0 + ti) X^{\sigma_0 + ti} \; i \; dt$$

Definition 3.5.4 (I)**.**

$$I_2(\nu, \epsilon, X, T, \sigma_1) := \frac{1}{2\pi i} \int_{\sigma_1}^{\sigma_0} \left(\frac{-\zeta'}{\zeta}(\sigma - iT) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma - iT) X^{\sigma - iT} \; d\sigma$$

Definition 3.5.5 (I)**.**

$$I_{37}(\nu, \epsilon, X, T, \sigma_1) := \frac{1}{2\pi i} \int_{-T}^{T} \left(\frac{-\zeta'}{\zeta}(\sigma_1 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_1 + ti) X^{\sigma_1 + ti} \; i \; dt$$

Definition 3.5.6 (I)**.**

$$I_8(\nu, \epsilon, X, T, \sigma_1) := \frac{1}{2\pi i} \int_{\sigma_1}^{\sigma_0} \left(\frac{-\zeta'}{\zeta}(\sigma + Ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma + Ti) X^{\sigma + Ti} \; d\sigma$$

Definition 3.5.7 (I)**.**

$$I_9(\nu, \epsilon, X, T) := \frac{1}{2\pi i} \int_{T}^{\infty} \left(\frac{-\zeta'}{\zeta}(\sigma_0 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_0 + ti) X^{\sigma_0 + ti} \; i \; dt$$

Definition 3.5.8 (I)**.**

$$I_3(\nu, \epsilon, X, T, \sigma_1) := \frac{1}{2\pi i} \int_{-T}^{-3} \left(\frac{-\zeta'}{\zeta}(\sigma_1 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_1 + ti) X^{\sigma_1 + ti} \; i \; dt$$

Definition 3.5.9 (I)**.**

$$I_7(\nu, \epsilon, X, T, \sigma_1) := \frac{1}{2\pi i} \int_{3}^{T} \left(\frac{-\zeta'}{\zeta}(\sigma_1 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_1 + ti) X^{\sigma_1 + ti} \; i \; dt$$

Definition 3.5.10 (I).

$$I_4(\nu, \epsilon, X, \sigma_1, \sigma_2) := \frac{1}{2\pi i} \int_{\sigma_2}^{\sigma_1} \left(\frac{-\zeta'}{\zeta}(\sigma - 3i) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma - 3i) X^{\sigma - 3i} \, d\sigma$$

Definition 3.5.11 (I).

$$I_6(\nu, \epsilon, X, \sigma_1, \sigma_2) := \frac{1}{2\pi i} \int_{\sigma_2}^{\sigma_1} \left(\frac{-\zeta'}{\zeta}(\sigma + 3i) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma + 3i) X^{\sigma + 3i} \, d\sigma$$

Definition 3.5.12 (I).

$$I_5(\nu, \epsilon, X, \sigma_2) := \frac{1}{2\pi i} \int_{-3}^{3} \left(\frac{-\zeta'}{\zeta}(\sigma_2 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_2 + ti) X^{\sigma_2 + ti} \, i \, dt$$

Lemma 3.5.3 (dlog-riemannZeta-bdd-on-vertical-lines)**.** For $\sigma_0 > 1$, there exists a constant $C > 0$ such that

$$\forall t \in \mathbb{R}, \quad \left\| \frac{\zeta'(\sigma_0 + ti)}{\zeta(\sigma_0 + ti)} \right\| \leq C.$$

Proof. Write as Dirichlet series and estimate trivially using Theorem 3.5.1. ☐

Lemma 3.5.4 (SmoothedChebyshevPull1-aux-integrable)**.** The integrand

$$\zeta'(s)/\zeta(s)\mathcal{M}(\widetilde{1_\epsilon})(s)X^s$$

is integrable on the contour $\sigma_0 + ti$ for $t \in \mathbb{R}$ and $\sigma_0 > 1$.

Proof. The $\zeta'(s)/\zeta(s)$ term is bounded, as is X^s, and the smoothing function $\mathcal{M}(\widetilde{1_\epsilon})(s)$ decays like $1/|s|^2$ by Theorem 3.3.10. Actually, we already know that $\mathcal{M}(\widetilde{1_\epsilon})(s)$ is integrable from Theorem 3.5.1, so we should just need to bound the rest. ☐

Lemma 3.5.5 (BddAboveOnRect)**.** Let $g : \mathbb{C} \to \mathbb{C}$ be a holomorphic function on a rectangle, then g is bounded above on the rectangle.

Proof. Use the compactness of the rectangle and the fact that holomorphic functions are continuous. ☐

Theorem 3.5.4 (SmoothedChebyshevPull1)**.** We have that

$$\psi_\epsilon(X) = \mathcal{M}(\widetilde{1_\epsilon})(1)X^1 + I_1 - I_2 + I_{37} + I_8 + I_9.$$

Proof. Pull rectangle contours and evaluate the pole at $s = 1$. ☐

Next pull contours to another box.

Lemma 3.5.6 (SmoothedChebyshevPull2)**.** We have that

$$I_{37} = I_3 - I_4 + I_5 + I_6 + I_7.$$

Proof. Mimic the proof of Lemma 3.5.4. ☐

We insert this information in ψ_ϵ. We add and subtract the integral over the box $[1 - \delta, 2] \times_{\mathbb{C}} [-T, T]$, which we evaluate as follows

Theorem 3.5.5 (ZetaBoxEval)**.** For all $\epsilon > 0$ sufficiently close to 0, the rectangle integral over $[1-\delta, 2] \times_{\mathbb{C}} [-T, T]$ of the integrand in ψ_ϵ is

$$\frac{X^1}{1}\mathcal{M}(\widetilde{1_\epsilon})(1) = X(1 + O(\epsilon)),$$

where the implicit constant is independent of X.

Proof. Unfold the definitions and apply Lemma 3.3.11. □

It remains to estimate all of the integrals.

This auxiliary lemma is useful for what follows.

Lemma 3.5.7 (IBound-aux1)**.** Given a natural number k and a real number $X_0 > 0$, there exists $C \geq 1$ so that for all $X \geq X_0$,

$$\log^k X \leq C \cdot X.$$

Proof. We use the fact that $\log^k X/X$ goes to 0 as $X \to \infty$. Then we use the extreme value theorem to find a constant C that works for all $X \geq X_0$. □

Lemma 3.5.8 (I1Bound)**.** We have that

$$|I_1(\nu, \epsilon, X, T)\ | \ll \frac{X}{\epsilon T}.$$

Same with I_9.

Proof. Unfold the definitions and apply the triangle inequality.

$$|I_1(\nu, \epsilon, X, T)| = \left| \frac{1}{2\pi i} \int_{-\infty}^{-T} \left(\frac{-\zeta'}{\zeta}(\sigma_0 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_0 + ti) X^{\sigma_0 + ti}\ i\ dt \right|$$

By Theorem 3.5.3 (once fixed!!), $\zeta'/\zeta(\sigma_0 + ti)$ is bounded by $\zeta'/\zeta(\sigma_0)$, and Theorem 3.4.14 gives $\ll 1/(\sigma_0 - 1)$ for the latter. This gives:

$$\leq \frac{1}{2\pi} \left| \int_{-\infty}^{-T} C \log X \cdot \frac{C'}{\epsilon|\sigma_0 + ti|^2} X^{\sigma_0}\ dt \right|,$$

where we used Theorem 3.3.10. Continuing the calculation, we have

$$\leq \log X \cdot C'' \frac{X^{\sigma_0}}{\epsilon} \int_{-\infty}^{-T} \frac{1}{t^2}\ dt\ \leq\ C''' \frac{X \log X}{\epsilon T},$$

where we used that $\sigma_0 = 1 + 1/\log X$, and $X^{\sigma_0} = X \cdot X^{1/\log X} = e \cdot X$. □

Lemma 3.5.9 (I2Bound)**.** Assuming a bound of the form of Lemma 3.4.28 we have that

$$|I_2(\nu, \epsilon, X, T)| \ll \frac{X}{\epsilon T}.$$

Proof. Unfold the definitions and apply the triangle inequality.

$$|I_2(\nu, \epsilon, X, T, \sigma_1)| = \left| \frac{1}{2\pi i} \int_{\sigma_1}^{\sigma_0} \left(\frac{-\zeta'}{\zeta}(\sigma - Ti) \right) \cdot \mathcal{M}(\tilde{1}_\epsilon)(\sigma - Ti) \cdot X^{\sigma - Ti} \, d\sigma \right|$$

$$\leq \frac{1}{2\pi} \int_{\sigma_1}^{\sigma_0} C \cdot \log T^9 \frac{C'}{\epsilon|\sigma - Ti|^2} X^{\sigma_0} \, d\sigma \leq C'' \cdot \frac{X \log T^9}{\epsilon T^2},$$

where we used Theorems 3.3.10, the hypothesised bound on zeta and the fact that $X^\sigma \leq X^{\sigma_0} = X \cdot X^{1/\log X} = e \cdot X$. Since $T > 3$, we have $\log T^9 \leq C''' T$. □

Lemma 3.5.10 (I8I2)**.** Symmetry between I_2 and I_8:

$$I_8(\nu, \epsilon, X, T) = -\overline{I_2(\nu, \epsilon, X, T)}.$$

Proof. This is a direct consequence of the definitions of I_2 and I_8. □

Lemma 3.5.11 (I8Bound)**.** We have that

$$|I_8(\nu, \epsilon, X, T)| \ll \frac{X}{\epsilon T}.$$

Proof. We deduce this from the corresponding bound for I_2, using the symmetry between I_2 and I_8. □

Lemma 3.5.12 (log-pow-over-xsq-integral-bounded)**.** For every n there is some absolute constant $C > 0$ such that

$$\int_3^T \frac{(\log x)^9}{x^2} dx < C$$

Proof. Induct on n and just integrate by parts. □

Lemma 3.5.13 (I3Bound)**.** Assuming a bound of the form of Lemma 3.4.28 we have that

$$|I_3(\nu, \epsilon, X, T)| \ll \frac{X}{\epsilon} X^{-\frac{A}{(\log T)^9}}.$$

Same with I_7.

Proof. Unfold the definitions and apply the triangle inequality.

$$|I_3(\nu, \epsilon, X, T, \sigma_1)| = \left| \frac{1}{2\pi i} \int_{-T}^{3} \left(\frac{-\zeta'}{\zeta}(\sigma_1 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_1 + ti) X^{\sigma_1 + ti} \, i \, dt \right|$$

$$\leq \frac{1}{2\pi} \int_{-T}^{3} C \cdot \log t^9 \frac{C'}{\epsilon|\sigma_1 + ti|^2} X^{\sigma_1} \, dt,$$

where we used Theorems 3.3.10 and the hypothesised bound on zeta. Now we estimate $X^{\sigma_1} = X \cdot X^{-A/\log T^9}$, and the integral is absolutely bounded. □

Lemma 3.5.14 (I4Bound)**.** We have that

$$|I_4(\nu, \epsilon, X, \sigma_1, \sigma_2)| \ll \frac{X}{\epsilon} X^{-\frac{A}{(\log T)^9}}.$$

Same with I_6.

Proof. The analysis of I_4 is similar to that of I_2, (in Lemma 3.5.9) but even easier. Let C be the sup of $-\zeta'/\zeta$ on the curve $\sigma_2 + 3i$ to $1 + 3i$ (this curve is compact, and away from the pole at $s = 1$). Apply Theorem 3.3.10 to get the bound $1/(\epsilon|s|^2)$, which is bounded by C'/ϵ. And X^s is bounded by $X^{\sigma_1} = X \cdot X^{-A/\log T^9}$. Putting these together gives the result. □

Lemma 3.5.15 (I5Bound)**.** We have that

$$|I_5(\nu, \epsilon, X, \sigma_2)| \ll \frac{X^{\sigma_2}}{\epsilon}.$$

Proof. Here ζ'/ζ is absolutely bounded on the compact interval $\sigma_2 + i[-3, 3]$, and X^s is bounded by X^{σ_2}. Using Theorem 3.3.10 gives the bound $1/(\epsilon|s|^2)$, which is bounded by C'/ϵ. Putting these together gives the result. □

3.6 MediumPNT

Theorem 3.6.1 (MediumPNT)**.** We have

$$\sum_{n \le x} \Lambda(n) = x + O(x \exp(-c(\log x)^{1/10})).$$

Proof. Evaluate the integrals. □

Chapter 4

Third Approach

4.1 Hadamard factorization

In this file, we prove the Hadamard Factorization theorem for functions of finite order, and prove that the zeta function is such.

4.2 Hoffstein-Lockhart

In this file, we use the Hoffstein-Lockhart construction to prove a zero-free region for zeta.

Hoffstein-Lockhart + Goldfeld-Hoffstein-Liemann

Instead of the "slick" identity $3+4\cos\theta+\cos 2\theta = 2(\cos\theta+1)^2 \geq 0$, we use the following more robust identity.

Theorem 4.2.1. For any $p > 0$ and $t \in \mathbb{R}$,

$$3 + p^{2it} + p^{-2it} + 2p^{it} + 2p^{-it} \geq 0.$$

Proof. This follows immediately from the identity

$$|1 + p^{it} + p^{-it}|^2 = 1 + p^{2it} + p^{-2it} + 2p^{it} + 2p^{-it} + 2.$$

□

[Note: identities of this type will work in much greater generality, especially for higher degree L-functions.]

This means that, for fixed t, we define the following alternate function.

Definition 4.2.1. For $\sigma > 1$ and $t \in \mathbb{R}$, define

$$F(\sigma) := \zeta^3(\sigma)\zeta^2(\sigma+it)\zeta^2(\sigma-it)\zeta(\sigma+2it)\zeta(\sigma-2it).$$

Theorem 4.2.2. Then F is real-valued, and whence $F(\sigma) \geq 1$ there.

Proof. That $\log F(\sigma) \geq 0$ for $\sigma > 1$ follows from Theorem 4.2.1. □

[Note: I often prefer to avoid taking logs of functions that, even if real-valued, have to be justified as being such. Instead, I like to start with "logF" as a convergent Dirichlet series, show that it is real-valued and non-negative, and then exponentiate...]

From this and Hadamard factorization, we deduce the following.

Theorem 4.2.3. There is a constant $c > 0$, so that $\zeta(s)$ does not vanish in the region $\sigma > 1 - \frac{c}{\log t}$, and moreover,

$$-\frac{\zeta'}{\zeta}(\sigma + it) \ll (\log t)^2$$

there.

Proof. Use Theorem 4.2.2 and Hadamard factorization. □

This allows us to quantify precisely the relationship between T and δ in Theorem 3.4.25....

4.3 Strong PNT

Definition 4.3.1. Given a complex function f, we define the function

$$g(z) := \begin{cases} \frac{f(z)}{z}, & z \neq 0; \\ f'(0), & z = 0. \end{cases}$$

Lemma 4.3.1. Let f be a complex function and let $z \neq 0$. Then, with g defined as in Definition 4.3.1,

$$g(z) = \frac{f(z)}{z}.$$

Proof. This follows directly from the definition of g. □

Lemma 4.3.2. Let f be a complex function analytic on an open set s containing 0 such that $f(0) = 0$. Then, with g defined as in Definition 4.3.1, g is analytic on s.

Proof. We need to show that g is complex differentiable at every point in s. For $z \neq 0$, this follows directly from the definition of g and the fact that f is analytic on s. For $z = 0$, we use the definition of the derivative and the fact that $f(0) = 0$:

$$\lim_{z\to 0} \frac{g(z) - g(0)}{z - 0} = \lim_{z\to 0} \frac{\frac{f(z)}{z} - f'(0)}{z} = \lim_{z\to 0} \frac{f(z) - f'(0)z}{z^2} = \lim_{z\to 0} \frac{f(z) - f(0) - f'(0)(z - 0)}{(z - 0)^2} = 0,$$

where the last equality follows from the definition of the derivative of f at 0. Thus, g is complex differentiable at 0 with derivative 0, completing the proof. □

Lemma 4.3.3. Let f be a complex function analytic on the closed ball $|z| \le R$ such that $f(0) = 0$. Then, with g defined as in Definition 4.3.1, g is analytic on $|z| \le R$.

Proof. The proof is similar to that of Lemma 4.3.2, but we need to consider two cases: when x is on the boundary of the closed ball and when it is in the interior. In the first case, we take a small open ball around x that lies entirely within the closed ball, and apply Lemma 4.3.2 on this smaller ball. In the second case, we can take the entire open ball centered at 0 with radius R, and again apply Lemma 4.3.2. In both cases, we use the fact that $f(0) = 0$ to ensure that the removable singularity at 0 is handled correctly. □

Definition 4.3.2. Given a complex function f and a real number M, we define the function

$$f_M(z) := \frac{g(z)}{2M - f(z)},$$

where g is defined as in Definition 4.3.1.

Lemma 4.3.4. Let $M > 0$. Let f be analytic on the closed ball $|z| \le R$ such that $f(0) = 0$ and suppose that $2M - f(z) \neq 0$ for all $|z| \le R$. Then, with f_M defined as in Definition 4.3.2, f_M is analytic on $|z| \le R$.

Proof. This follows directly from Lemma 4.3.3 and the fact that the difference of two analytic functions is analytic. □

Lemma 4.3.5. Let $M > 0$ and let x be a complex number such that $\Re x \le M$. Then, $|x| \le |2M - x|$.

Proof. We square both sides and simplify to obtain the equivalent inequality

$$0 \le 4M^2 - 4M\Re x,$$

which follows directly from the assumption $\Re x \le M$ and the positivity of M. □

Theorem 4.3.1 (borelCaratheodory-closedBall)**.** Let R, $M > 0$. Let f be analytic on $|z| \le R$ such that $f(0) = 0$ and suppose $\Re f(z) \le M$ for all $|z| \le R$. Then for any $0 < r < R$,

$$\sup_{|z| \le r} |f(z)| \le \frac{2Mr}{R - r}.$$

Proof. Let

$$f_M(z) = \frac{f(z)/z}{2M - f(z)}.$$

Note that $2M - f(z) \neq 0$ because $\Re(2M - f(z)) = 2M - \Re f(z) \ge M > 0$. Additionally, since $f(z)$ has a zero at 0, we know that $f(z)/z$ is analytic on $|z| \le R$. Likewise, $f_M(z)$ is analytic on $|z| \le R$.

Now note that $|f(z)| \le |2M - f(z)|$ since $\Re f(z) \le M$. Thus we have that

$$|f_M(z)| = \frac{|f(z)|/|z|}{|2M - f(z)|} \le \frac{1}{|z|}.$$

Now by the maximum modulus principle, we know the maximum of $|f_M|$ must occur on the boundary where $|z| = R$. Thus, $|f_M(z)| \le 1/R$ for all $|z| \le R$. So for $|z| = r$ we have

$$|f_M(z)| = \frac{|f(z)|/r}{|2M - f(z)|} \le \frac{1}{R} \implies R\,|f(z)| \le r\,|2M - f(z)| \le 2Mr + r\,|f(z)|.$$

Which by algebraic manipulation gives

$$|f(z)| \le \frac{2Mr}{R - r}.$$

Once more, by the maximum modulus principle, we know the maximum of $|f|$ must occur on the boundary where $|z| = r$. Thus, the desired result immediately follows □

Lemma 4.3.6 (DerivativeBound)**.** Let R, $M > 0$ and $0 < r < r' < R$. Let f be analytic on $|z| \le R$ such that $f(0) = 0$ and suppose $\Re f(z) \le M$ for all $|z| \le R$. Then we have that

$$|f'(z)| \le \frac{2M(r')^2}{(R - r')(r' - r)^2}$$

for all $|z| \le r$.

Proof. By Lemma 4.3.7 we know that

$$f'(z) = \frac{1}{2\pi i} \oint_{|w|=r'} \frac{f(w)}{(w-z)^2}\, dw = \frac{1}{2\pi} \int_0^{2\pi} \frac{r' e^{it} f(r' e^{it})}{(r' e^{it} - z)^2}\, dt.$$

Thus,

$$|f'(z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{r' e^{it} f(r' e^{it})}{(r' e^{it} - z)^2}\, dt \right| \le \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{r' e^{it} f(r' e^{it})}{(r' e^{it} - z)^2} \right| dt. \tag{4.1}$$

Now applying Theorem **??**, and noting that $r' - r \le |r' e^{it} - z|$, we have that

$$\left| \frac{r' e^{it} f(r' e^{it})}{(r' e^{it} - z)^2} \right| \le \frac{2M(r')^2}{(R - r')(r' - r)^2}.$$

Substituting this into Equation (4.2) and evaluating the integral completes the proof. ☐

This upstreamed from https://github.com/math-inc/strongpnt/tree/main

Lemma 4.3.7 (cauchy-formula-deriv)**.** Let f be analytic on $|z| \le R$. For any z with $|z| \le r$ and any r' with $0 < r < r' < R$ we have

$$f'(z) = \frac{1}{2\pi i} \oint_{|w|=r'} \frac{f(w)}{(w-z)^2}\, dw = \frac{1}{2\pi} \int_0^{2\pi} \frac{r' e^{it} f(r' e^{it})}{(r' e^{it} - z)^2}\, dt.$$

Proof. This is just Cauchy's integral formula for derivatives. ☐

Lemma 4.3.8 (DerivativeBound)**.** Let R, $M > 0$ and $0 < r < r' < R$. Let f be analytic on $|z| \le R$ such that $f(0) = 0$ and suppose $\Re f(z) \le M$ for all $|z| \le R$. Then we have that

$$|f'(z)| \le \frac{2M(r')^2}{(R - r')(r' - r)^2}$$

for all $|z| \le r$.

Proof. By Lemma 4.3.7 we know that

$$f'(z) = \frac{1}{2\pi i} \oint_{|w|=r'} \frac{f(w)}{(w-z)^2}\, dw = \frac{1}{2\pi} \int_0^{2\pi} \frac{r' e^{it} f(r' e^{it})}{(r' e^{it} - z)^2}\, dt.$$

Thus,

$$|f'(z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{r' e^{it} f(r' e^{it})}{(r' e^{it} - z)^2}\, dt \right| \le \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{r' e^{it} f(r' e^{it})}{(r' e^{it} - z)^2} \right| dt. \tag{4.2}$$

Now applying Theorem **??**, and noting that $r' - r \le |r' e^{it} - z|$, we have that

$$\left| \frac{r' e^{it} f(r' e^{it})}{(r' e^{it} - z)^2} \right| \le \frac{2M(r')^2}{(R - r')(r' - r)^2}.$$

Substituting this into Equation (4.2) and evaluating the integral completes the proof. ☐

Theorem 4.3.2 (BorelCaratheodoryDeriv)**.** Let R, $M > 0$. Let f be analytic on $|z| \leq R$ such that $f(0) = 0$ and suppose $\Re f(z) \leq M$ for all $|z| \leq R$. Then for any $0 < r < R$,

$$|f'(z)| \leq \frac{16MR^2}{(R-r)^3}$$

for all $|z| \leq r$.

Proof. Using Lemma 4.3.8 with $r' = (R+r)/2$, and noting that $r < R$, we have that

$$|f'(z)| \leq \frac{4M(R+r)^2}{(R-r)^3} \leq \frac{16MR^2}{(R-r)^3}.$$

□

Theorem 4.3.3 (LogOfAnalyticFunction)**.** Let $0 < r < R < 1$. Let $B : \overline{\mathbb{D}_R} \to \mathbb{C}$ be analytic on neighborhoods of points in $\overline{\mathbb{D}_R}$ with $B(z) \neq 0$ for all $z \in \overline{\mathbb{D}_R}$. Then there exists $J_B : \overline{\mathbb{D}_r} \to \mathbb{C}$ that is analytic on neighborhoods of points in $\overline{\mathbb{D}_r}$ such that

- $J_B(0) = 0$
- $J_B'(z) = B'(z)/B(z)$
- $\log|B(z)| - \log|B(0)| = \Re J_B(z)$

for all $z \in \overline{\mathbb{D}_r}$.

Proof. We let $J_B(z) = \operatorname{Log} B(z) - \operatorname{Log} B(0)$. Then clearly, $J_B(0) = 0$ and $J_B'(z) = B'(z)/B(z)$. Showing the third property is a little more difficult, but by no standards terrible. Exponentiating $J_B(z)$ we have that

$$\exp(J_B(z)) = \exp(\operatorname{Log} B(z) - \operatorname{Log} B(0)) = \frac{B(z)}{B(0)} \implies B(z) = B(0)\exp(J_B(z)).$$

Now taking the modulus

$$|B(z)| = |B(0)| \cdot |\exp(J_B(z))| = |B(0)| \cdot \exp(\Re J_B(z)).$$

Taking the real logarithm of both sides and rearranging gives the third point. □

Definition 4.3.3 (SetOfZeros)**.** Let $R > 0$ and $f : \overline{\mathbb{D}_R} \to \mathbb{C}$. Define the set of zeros $\mathcal{K}_f(R) = \{\rho \in \mathbb{C} : |\rho| \leq R,\ f(\rho) = 0\}$.

Definition 4.3.4 (ZeroOrder)**.** Let $0 < R < 1$ and $f : \mathbb{C} \to \mathbb{C}$ be analtyic on neighborhoods of points in $\overline{\mathbb{D}_1}$. For any zero $\rho \in \mathcal{K}_f(R)$, we define $m_f(\rho)$ as the order of the zero ρ w.r.t f.

Lemma 4.3.9 (ZeroFactorization)**.** Let $f : \overline{\mathbb{D}_1} \to \mathbb{C}$ be analytic on neighborhoods of points in $\overline{\mathbb{D}_1}$ with $f(0) \neq 0$. For all $\rho \in \mathcal{K}_f(1)$ there exists $h_\rho(z)$ that is analytic at ρ, $h_\rho(\rho) \neq 0$, and $f(z) = (z-\rho)^{m_f(\rho)}\, h_\rho(z)$.

Proof. Since f is analytic on neighborhoods of points in $\overline{\mathbb{D}_1}$ we know that there exists a series expansion about ρ:

$$f(z) = \sum_{0 \leq n} a_n\,(z-\rho)^n.$$

Now if we let m be the smallest number such that $a_m \neq 0$, then

$$f(z) = \sum_{0 \le n} a_n \, (z-\rho)^n = \sum_{m \le n} a_n \, (z-\rho)^n = (z-\rho)^m \sum_{m \le n} a_n \, (z-\rho)^{n-m} = (z-\rho)^m \, h_\rho(z).$$

Trivially, $h_\rho(z)$ is analytic at ρ (we have written down the series expansion); now note that

$$h_\rho(\rho) = \sum_{m \le n} a_n (\rho-\rho)^{n-m} = \sum_{m \le n} a_n 0^{n-m} = a_m \neq 0.$$

□

Definition 4.3.5 (CFunction)**.** Let $0 < r < R < 1$, and $f : \overline{\mathbb{D}_1} \to \mathbb{C}$ be analytic on neighborhoods of points in $\overline{\mathbb{D}_1}$ with $f(0) \neq 0$. We define a function $C_f : \overline{\mathbb{D}_R} \to \mathbb{C}$ as follows. This function is constructed by dividing $f(z)$ by a polynomial whose roots are the zeros of f inside $\overline{\mathbb{D}_r}$.

$$C_f(z) = \begin{cases} \dfrac{f(z)}{\prod_{\rho \in \mathcal{K}_f(r)} (z-\rho)^{m_f(\rho)}} & \text{for } z \notin \mathcal{K}_f(r) \\ \dfrac{h_z(z)}{\prod_{\rho \in \mathcal{K}_f(r) \setminus \{z\}} (z-\rho)^{m_f(\rho)}} & \text{for } z \in \mathcal{K}_f(r) \end{cases}$$

where $h_z(z)$ comes from Lemma 4.3.9.

Definition 4.3.6 (BlaschkeB)**.** Let $0 < r < R < 1$, and $f : \overline{\mathbb{D}_1} \to \mathbb{C}$ be analytic on neighborhoods of points in $\overline{\mathbb{D}_1}$ with $f(0) \neq 0$. We define a function $B_f : \overline{\mathbb{D}_R} \to \mathbb{C}$ as follows.

$$B_f(z) = C_f(z) \prod_{\rho \in \mathcal{K}_f(r)} \left(R - \frac{z\overline{\rho}}{R} \right)^{m_f(\rho)}$$

Lemma 4.3.10 (BlaschkeOfZero)**.** Let $0 < r < R < 1$, and $f : \overline{\mathbb{D}_1} \to \mathbb{C}$ be analytic on neighborhoods of points in $\overline{\mathbb{D}_1}$ with $f(0) \neq 0$. Then

$$|B_f(0)| = |f(0)| \prod_{\rho \in \mathcal{K}_f(r)} \left(\frac{R}{|\rho|} \right)^{m_f(\rho)}.$$

Proof. Since $f(0) \neq 0$, we know that $0 \notin \mathcal{K}_f(r)$. Thus,

$$C_f(0) = \frac{f(0)}{\prod_{\rho \in \mathcal{K}_f(r)} (-\rho)^{m_f(\rho)}}.$$

Thus, substituting this into Definition 4.3.6,

$$|B_f(0)| = |C_f(0)| \prod_{\rho \in \mathcal{K}_f(r)} R^{m_f(\rho)} = |f(0)| \prod_{\rho \in \mathcal{K}_f(r)} \left(\frac{R}{|\rho|} \right)^{m_f(\rho)}.$$

□

Lemma 4.3.11 (DiskBound)**.** Let $B > 1$ and $0 < R < 1$. If $f : \mathbb{C} \to \mathbb{C}$ is a function analytic on neighborhoods of points in $\overline{\mathbb{D}_1}$ with $|f(z)| \le B$ for $|z| \le R$, then $|B_f(z)| \le B$ for $|z| \le R$ also.

Proof. For $|z| = R$, we know that $z \notin \mathcal{K}_f(r)$. Thus,

$$C_f(z) = \frac{f(z)}{\prod_{\rho \in \mathcal{K}_f(r)} (z-\rho)^{m_f(\rho)}}.$$

Thus, substituting this into Definition 4.3.6,

$$|B_f(z)| = |f(z)| \prod_{\rho \in \mathcal{K}_f(r)} \left| \frac{R - z\overline{\rho}/R}{z-\rho} \right|^{m_f(\rho)}.$$

But note that

$$\left| \frac{R - z\overline{\rho}/R}{z-\rho} \right| = \frac{|R^2 - z\overline{\rho}|/R}{|z-\rho|} = \frac{|z| \cdot |\overline{z-\rho}|/R}{|z-\rho|} = 1.$$

So we have that $|B_f(z)| = |f(z)| \leq B$ when $|z| = R$. Now by the maximum modulus principle, we know that the maximum of $|B_f|$ must occur on the boundary where $|z| = R$. Thus $|B_f(z)| \leq B$ for all $|z| \leq R$. ☐

Theorem 4.3.4 (ZerosBound)**.** Let $B > 1$ and $0 < r < R < 1$. If $f : \mathbb{C} \to \mathbb{C}$ is a function analytic on neighborhoods of points in $\overline{\mathbb{D}_1}$ with $f(0) = 1$ and $|f(z)| \leq B$ for $|z| \leq R$, then

$$\sum_{\rho \in \mathcal{K}_f(r)} m_f(\rho) \leq \frac{\log B}{\log(R/r)}.$$

Proof. Since $f(0) = 1$, we know that $0 \notin \mathcal{K}_f(r)$. Thus,

$$C_f(0) = \frac{f(0)}{\prod_{\rho \in \mathcal{K}_f(r)} (-\rho)^{m_f(\rho)}}.$$

Thus, substituting this into Definition 4.3.6,

$$(R/r)^{\sum_{\rho \in \mathcal{K}_f(r)} m_f(\rho)} = \prod_{\rho \in \mathcal{K}_f(r)} \left(\frac{R}{r}\right)^{m_f(\rho)} \leq \prod_{\rho \in \mathcal{K}_f(r)} \left(\frac{R}{|\rho|}\right)^{m_f(\rho)} = |B_f(0)| \leq B$$

whereby Lemma 4.3.11 we know that $|B_f(z)| \leq B$ for all $|z| \leq R$. Taking the logarithm of both sides and rearranging gives the desired result. ☐

Definition 4.3.7 (JBlaschke)**.** Let $B > 1$ and $0 < R < 1$. If $f : \mathbb{C} \to \mathbb{C}$ is a function analytic on neighborhoods of points in $\overline{\mathbb{D}_1}$ with $f(0) = 1$, define $L_f(z) = J_{B_f}(z)$ where J is from Theorem 4.3.3 and B_f is from Definition 4.3.6.

Lemma 4.3.12 (BlaschkeNonZero)**.** Let $0 < r < R < 1$ and $f : \overline{\mathbb{D}_1} \to \mathbb{C}$ be analytic on neighborhoods of points in $\overline{\mathbb{D}_1}$. Then $B_f(z) \neq 0$ for all $z \in \overline{\mathbb{D}_r}$.

Proof. Suppose that $z \in \mathcal{K}_f(r)$. Then we have that

$$C_f(z) = \frac{h_z(z)}{\prod_{\rho \in \mathcal{K}_f(r) \setminus \{z\}} (z-\rho)^{m_f(\rho)}}.$$

where $h_z(z) \neq 0$ according to Lemma 4.3.9. Thus, substituting this into Definition 4.3.6,

$$|B_f(z)| = |h_z(z)| \cdot \left|R - \frac{|z|^2}{R}\right|^{m_f(z)} \prod_{\rho \in \mathcal{K}_f(r) \setminus \{z\}} \left|\frac{R - z\overline{\rho}/R}{z - \rho}\right|^{m_f(\rho)}. \tag{4.3}$$

Trivially, $|h_z(z)| \neq 0$. Now note that

$$\left|R - \frac{|z|^2}{R}\right| = 0 \implies |z| = R.$$

However, this is a contradiction because $z \in \overline{\mathbb{D}_r}$ tells us that $|z| \leq r < R$. Similarly, note that

$$\left|\frac{R - z\overline{\rho}/R}{z - \rho}\right| = 0 \implies |z| = \frac{R^2}{|\overline{\rho}|}.$$

However, this is also a contradiction because $\rho \in \mathcal{K}_f(r)$ tells us that $R < R^2/|\overline{\rho}| = |z|$, but $z \in \overline{\mathbb{D}_r}$ tells us that $|z| \leq r < R$. So, we know that

$$\left|R - \frac{|z|^2}{R}\right| \neq 0 \qquad \text{and} \qquad \left|\frac{R - z\overline{\rho}/R}{z - \rho}\right| \neq 0 \quad \text{for all} \quad \rho \in \mathcal{K}_f(r) \setminus \{z\}.$$

Applying this to Equation (4.3) we have that $|B_f(z)| \neq 0$. So, $B_f(z) \neq 0$.

Now suppose that $z \notin \mathcal{K}_f(r)$. Then we have that

$$C_f(z) = \frac{f(z)}{\prod_{\rho \in \mathcal{K}_f(r)} (z - \rho)^{m_f(\rho)}}.$$

Thus, substituting this into Definition 4.3.6,

$$|B_f(z)| = |f(z)| \prod_{\rho \in \mathcal{K}_f(r)} \left|\frac{R - z\overline{\rho}/R}{z - \rho}\right|^{m_f(\rho)}. \tag{4.4}$$

We know that $|f(z)| \neq 0$ since $z \notin \mathcal{K}_f(r)$. Now note that

$$\left|\frac{R - z\overline{\rho}/R}{z - \rho}\right| = 0 \implies |z| = \frac{R^2}{|\overline{\rho}|}.$$

However, this is a contradiction because $\rho \in \mathcal{K}_f(r)$ tells us that $R < R^2/|\overline{\rho}| = |z|$, but $z \in \overline{\mathbb{D}_r}$ tells us that $|z| \leq r < R$. So, we know that

$$\left|\frac{R - z\overline{\rho}/R}{z - \rho}\right| \neq 0 \quad \text{for all} \quad \rho \in \mathcal{K}_f(r).$$

Applying this to Equation (4.4) we have that $|B_f(z)| \neq 0$. So, $B_f(z) \neq 0$.

We have shown that $B_f(z) \neq 0$ for both $z \in \mathcal{K}_f(r)$ and $z \notin \mathcal{K}_f(r)$, so the result follows. □

Theorem 4.3.5 (JBlaschkeDerivBound)**.** Let $B > 1$ and $0 < r' < r < R < 1$. If $f : \mathbb{C} \to \mathbb{C}$ is a function analytic on neighborhoods of points in $\overline{\mathbb{D}_1}$ with $f(0) = 1$ and $|f(z)| \leq B$ for all $|z| \leq R$, then for all $|z| \leq r'$

$$|L'_f(z)| \leq \frac{16 \log(B)\, r^2}{(r - r')^3}$$

Proof. By Lemma 4.3.11 we immediately know that $|B_f(z)| \leq B$ for all $|z| \leq R$. Now since $L_f = J_{B_f}$ by Definition 4.3.7, by Theorem 4.3.3 we know that

$$L_f(0) = 0 \qquad \text{and} \qquad \Re L_f(z) = \log|B_f(z)| - \log|B_f(0)| \leq \log|B_f(z)| \leq \log B$$

for all $|z| \leq r$. So by Theorem 4.3.2, it follows that

$$|L'_f(z)| \leq \frac{16\log(B)\, r^2}{(r-r')^3}$$

for all $|z| \leq r'$. □

Theorem 4.3.6 (FinalBound)**.** Let $B > 1$ and $0 < r' < r < R' < R < 1$. If $f : \mathbb{C} \to \mathbb{C}$ is a function analytic on neighborhoods of points in $\overline{\mathbb{D}_1}$ with $f(0) = 1$ and $|f(z)| \leq B$ for all $|z| \leq R$, then for all $z \in \overline{\mathbb{D}_{R'}} \setminus \mathcal{K}_f(R')$ we have

$$\left|\frac{f'}{f}(z) - \sum_{\rho\in\mathcal{K}_f(R')} \frac{m_f(\rho)}{z-\rho}\right| \leq \left(\frac{16r^2}{(r-r')^3} + \frac{1}{(R^2/R' - R')\,\log(R/R')}\right)\log B.$$

Proof. Since $z \in \overline{\mathbb{D}_{r'}} \setminus \mathcal{K}_f(R')$ we know that $z \notin \mathcal{K}_f(R')$; thus, by Definition 4.3.5 we know that

$$C_f(z) = \frac{f(z)}{\displaystyle\prod_{\rho\in\mathcal{K}_f(R')} (z-\rho)^{m_f(\rho)}}.$$

Substituting this into Definition 4.3.6 we have that

$$B_f(z) = f(z) \prod_{\rho\in\mathcal{K}_f(R')} \left(\frac{R - z\overline{\rho}/R}{z-\rho}\right)^{m_f(\rho)}.$$

Taking the complex logarithm of both sides we have that

$$\operatorname{Log} B_f(z) = \operatorname{Log} f(z) + \sum_{\rho\in\mathcal{K}_f(R')} m_f(\rho)\operatorname{Log}(R - z\overline{\rho}/R) - \sum_{\rho\in\mathcal{K}_f(R')} m_f(\rho)\operatorname{Log}(z-\rho).$$

Taking the derivative of both sides we have that

$$\frac{B'_f}{B_f}(z) = \frac{f'}{f}(z) + \sum_{\rho\in\mathcal{K}_f(R')} \frac{m_f(\rho)}{z - R^2/\rho} - \sum_{\rho\in\mathcal{K}_f(R')} \frac{m_f(\rho)}{z-\rho}.$$

By Definition 4.3.7 and Theorem 4.3.3 we recall that

$$L_f(z) = J_{B_f}(z) = \operatorname{Log} B_f(z) - \operatorname{Log} B_f(0).$$

Taking the derivative of both sides we have that $L'_f(z) = (B'_f/B_f)(z)$. Thus,

$$\frac{f'}{f}(z) - \sum_{\rho\in\mathcal{K}_f(R')} \frac{m_f(\rho)}{z-\rho} = L'_f(z) - \sum_{\rho\in\mathcal{K}_f(R')} \frac{m_f(\rho)}{z - R^2/\rho}.$$

Now since $z \in \overline{\mathbb{D}_{R'}}$ and $\rho \in \mathcal{K}_f(R')$, we know that $R^2/R' - R' \leq |z - R^2/\rho|$. Thus by the triangle inequality we have

$$\left| \frac{f'}{f}(z) - \sum_{\rho \in \mathcal{K}_f(R')} \frac{m_f(\rho)}{z - \rho} \right| \leq |L'_f(z)| + \left(\frac{1}{R^2/R' - R'} \right) \sum_{\rho \in \mathcal{K}_f(R')} m_f(\rho).$$

Now by Theorem 4.3.4 and 4.3.5 we get our desired result with a little algebraic manipulation. □

Theorem 4.3.7 (ZetaFixedLowerBound)**.** For all $t \in \mathbb{R}$ one has

$$|\zeta(3/2 + it)| \geq \frac{\zeta(3)}{\zeta(3/2)}.$$

Proof. From the Euler product expansion of ζ, we have that for $\Re s > 1$

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

Thus, we have that

$$\frac{\zeta(2s)}{\zeta(s)} = \prod_p \frac{1 - p^{-s}}{1 - p^{-2s}} = \prod_p \frac{1}{1 + p^{-s}}.$$

Now note that $|1 - p^{-(3/2+it)}| \leq 1 + |p^{-(3/2+it)}| = 1 + p^{-3/2}$. Thus,

$$|\zeta(3/2 + it)| = \prod_p \frac{1}{|1 - p^{-(3/2+it)}|} \geq \prod_p \frac{1}{1 + p^{-3/2}} = \frac{\zeta(3)}{\zeta(3/2)}$$

for all $t \in \mathbb{R}$ as desired. □

Lemma 4.3.13 (ZetaAltFormula)**.** Let

$$\zeta_0(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \{x\}\, x^{-s} \,\frac{dx}{x}.$$

We have that $\zeta(s) = \zeta_0(s)$ for $\sigma > 1$.

Proof. Note that for $\sigma > 1$ we have

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \sum_{n=1}^\infty \frac{n}{n^s} - \sum_{n=1}^\infty \frac{n-1}{n^s} = \sum_{n=1}^\infty \frac{n}{n^s} - \sum_{n=0}^\infty \frac{n}{(n+1)^s} = \sum_{n=1}^\infty \frac{n}{n^s} - \sum_{n=1}^\infty \frac{n}{(n+1)^s}.$$

Thus

$$\zeta(s) = \sum_{n=1}^\infty n\,(n^{-s} - (n+1)^{-s}).$$

Now we note that

$$s \int_n^{n+1} x^{-s} \,\frac{dx}{x} = s \left(-\frac{1}{s}\, x^{-s} \right)_n^{n+1} = n^{-s} - (n+1)^{-s}.$$

So, substituting this we have

$$\zeta(s) = \sum_{n=1}^{\infty} n\left(n^{-s} - (n+1)^{-s}\right) = s\sum_{n=1}^{\infty} n\int_n^{n+1} x^{-s}\,\frac{dx}{x} = s\int_1^{\infty} \lfloor x\rfloor\, x^{-s}\,\frac{dx}{x}.$$

But noting that $\lfloor x\rfloor = x - \{x\}$ we have that

$$\zeta(s) = s\int_1^{\infty} \lfloor x\rfloor\, x^{-s}\,\frac{dx}{x} = s\int_1^{\infty} x^{-s}\,dx - s\int_1^{\infty} \{x\}\, x^{-s}\,\frac{dx}{x}.$$

Evaluating the first integral completes the result. □

Lemma 4.3.14 (ZetaAltFormulaAnalytic)**.** We have that $\zeta_0(s)$ is analytic for all $s \in S$ where $S = \{s \in \mathbb{C} : \Re s > 0,\ s \neq 1\}$.

Proof. Note that we have

$$\left|\int_1^{\infty} \{x\}\, x^{-s}\,\frac{dx}{x}\right| \le \int_1^{\infty} \left|\{x\}\, x^{-s-1}\right| dx \le \int_1^{\infty} x^{-\sigma-1}\,dx = \frac{1}{\sigma}.$$

So this integral converges uniformly on compact subsets of S, which tells us that it is analytic on S. So it immediately follows that $\zeta_0(s)$ is analytic on S as well, since S avoids the pole at $s = 1$ coming from the $(s-1)^{-1}$ term. □

Lemma 4.3.15 (ZetaExtend)**.** We have that

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^{\infty} \{x\}\, x^{-s}\,\frac{dx}{x}$$

for all $s \in S$.

Proof. This is an immediate consequence of the identity theorem. □

Theorem 4.3.8 (GlobalBound)**.** For all $s \in \mathbb{C}$ with $|s| \le 1$ and $t \in \mathbb{R}$ with $|t| \ge 2$, we have that

$$|\zeta(s + 3/2 + it)| \le 7 + 2\,|t|.$$

Proof. For the sake of clearer proof writing let $z = s + 3/2 + it$. Since $|s| \le 1$ we know that $1/2 \le \Re z$; additionally, as $|t| \ge 2$, we know $1 \le |\Im z|$. So, $z \in S$. Thus, from Lemma 4.3.15 we know that

$$|\zeta(z)| \le 1 + \frac{1}{|z-1|} + |z| \cdot \left|\int_1^{\infty} \{x\}\, x^{-z}\,\frac{dx}{x}\right|$$

by applying the triangle inequality. Now note that $|z - 1| \ge 1$. Likewise,

$$|z| \cdot \left|\int_1^{\infty} \{x\}\, x^{-z}\,\frac{dx}{x}\right| \le |z| \int_1^{\infty} \left|\{x\}\, x^{-z-1}\right| dx \le |z| \int_1^{\infty} x^{-\Re z-1}\,dx = \frac{|z|}{\Re z} \le 2\,|z|.$$

Thus we have that,

$$|\zeta(s+3/2+it)| = |\zeta(z)| \le 1 + 1 + 2\,|z| = 2 + 2\,|s+3/2+it| \le 2 + 2\,|s| + 3 + 2\,|it| \le 7 + 2\,|t|.$$

□

Theorem 4.3.9 (LogDerivZetaFinalBound)**.** Let $t \in \mathbb{R}$ with $|t| \geq 2$ and $0 < r' < r < R' < R < 1$. If $f(z) = \zeta(z + 3/2 + it)$, then for all $z \in \overline{\mathbb{D}'_R} \setminus \mathcal{K}_f(R')$ we have that

$$\left| \frac{f'}{f}(z) - \sum_{\rho \in \mathcal{K}_f(R')} \frac{m_f(\rho)}{z - \rho} \right| \ll \left(\frac{16r^2}{(r - r')^3} + \frac{1}{(R^2/R' - R') \log(R/R')} \right) \log |t|.$$

Proof. Let $g(z) = \zeta(z + 3/2 + it)/\zeta(3/2 + it)$. Note that $g(0) = 1$ and for $|z| \leq R$

$$|g(z)| = \frac{|\zeta(z + 3/2 + it)|}{|\zeta(3/2 + it)|} \leq \frac{\zeta(3/2)}{\zeta(3)} \cdot (7 + 2\,|t|) \leq \frac{13\,\zeta(3/2)}{3\,\zeta(3)}\,|t|$$

by Theorems 4.3.7 and 4.3.8. Thus by Theorem 4.3.6 we have that

$$\left| \frac{g'}{g}(z) - \sum_{\rho \in \mathcal{K}_g(R')} \frac{m_g(\rho)}{z - \rho} \right| \leq \left(\frac{16r^2}{(r - r')^3} + \frac{1}{(R^2/R' - R') \log(R/R')} \right) \left(\log |t| + \log \left(\frac{13\,\zeta(3/2)}{3\,\zeta(3)} \right) \right).$$

Now note that $f'/f = g'/g$, $\mathcal{K}_f(R') = \mathcal{K}_g(R')$, and $m_g(\rho) = m_f(\rho)$ for all $\rho \in \mathcal{K}_f(R')$. Thus we have that,

$$\left| \frac{f'}{f}(z) - \sum_{\rho \in \mathcal{K}_f(R')} \frac{m_f(\rho)}{z - \rho} \right| \ll \left(\frac{16r^2}{(r - r')^3} + \frac{1}{(R^2/R' - R') \log(R/R')} \right) \log |t|$$

where the implied constant C is taken to be

$$C \geq 1 + \frac{\log((13\,\zeta(3/2))/(3\,\zeta(3)))}{\log 2}.$$

□

Definition 4.3.8 (ZeroWindows)**.** Let $\mathcal{Z}_t = \{\rho \in \mathbb{C} : \zeta(\rho) = 0,\ |\rho - (3/2 + it)| \leq 5/6\}$.

Lemma 4.3.16 (SumBoundI)**.** For all $\delta \in (0, 1)$ and $t \in \mathbb{R}$ with $|t| \geq 2$ we have

$$\left| \frac{\zeta'}{\zeta}(1 + \delta + it) - \sum_{\rho \in \mathcal{Z}_t} \frac{m_\zeta(\rho)}{1 + \delta + it - \rho} \right| \ll \log |t|.$$

Proof. We apply Theorem 4.3.9 where $r' = 2/3$, $r = 3/4$, $R' = 5/6$, and $R = 8/9$. Thus, for all $z \in \overline{\mathbb{D}_{5/6}} \setminus \mathcal{K}_f(5/6)$ we have that

$$\left| \frac{\zeta'}{\zeta}(z + 3/2 + it) - \sum_{\rho \in \mathcal{K}_f(5/6)} \frac{m_f(\rho)}{z - \rho} \right| \ll \log |t|$$

where $f(z) = \zeta(z + 3/2 + it)$ for $t \in \mathbb{R}$ with $|t| \geq 3$. Now if we let $z = -1/2 + \delta$, then $z \in (-1/2, 1/2) \subseteq \overline{\mathbb{D}_{5/6}}$. Additionally, $f(z) = \zeta(1 + \delta + it)$, where $1 + \delta + it$ lies in the zero-free region where $\sigma > 1$. Thus, $z \notin \mathcal{K}_f(5/6)$. So,

$$\left| \frac{\zeta'}{\zeta}(1 + \delta + it) - \sum_{\rho \in \mathcal{K}_f(5/6)} \frac{m_f(\rho)}{-1/2 + \delta - \rho} \right| \ll \log |t|.$$

But now note that if $\rho \in \mathcal{K}_f(5/6)$, then $\zeta(\rho+3/2+it) = 0$ and $|\rho| \le 5/6$. Thus, $\rho+3/2+it \in \mathcal{Z}_t$. Additionally, note that $m_f(\rho) = m_\zeta(\rho + 3/2 + it)$. So changing variables using these facts gives us that

$$\left| \frac{\zeta'}{\zeta}(1+\delta+it) - \sum_{\rho \in \mathcal{Z}_t} \frac{m_\zeta(\rho)}{1+\delta+it-\rho} \right| \ll \log|t|.$$

□

Lemma 4.3.17 (ShiftTwoBound)**.** For all $\delta \in (0,1)$ and $t \in \mathbb{R}$ with $|t| \ge 2$ we have

$$-\mathfrak{R}\left(\frac{\zeta'}{\zeta}(1+\delta+2it)\right) \ll \log|t|.$$

Proof. Note that, for $\rho \in \mathcal{Z}_{2t}$

$$\begin{aligned}\mathfrak{R}\left(\frac{1}{1+\delta+2it-\rho}\right) &= \mathfrak{R}\left(\frac{1+\delta-2it-\overline{\rho}}{(1+\delta+2it-\rho)(1+\delta-2it-\overline{\rho})}\right) \\ &= \frac{\mathfrak{R}(1+\delta-2it-\overline{\rho})}{|1+\delta+2it-\rho|^2} = \frac{1+\delta-\mathfrak{R}\rho}{(1+\delta-\mathfrak{R}\rho)^2+(2t-\mathfrak{I}\rho)^2}.\end{aligned}$$

Now since $\rho \in \mathcal{Z}_{2t}$, we have that $|\rho - (3/2+2it)| \le 5/6$. So, we have $\mathfrak{R}\rho \in (2/3, 7/3)$ and $\mathfrak{I}\rho \in (2t-5/6, 2t+5/6)$. Thus, we have that

$$1/3 < 1+\delta-\mathfrak{R}\rho \qquad \text{and} \qquad (1+\delta-\mathfrak{R}\rho)^2+(2t-\mathfrak{I}\rho)^2 < 16/9+25/36 = 89/36.$$

Which implies that

$$0 \le \frac{12}{89} < \frac{1+\delta-\mathfrak{R}\rho}{(1+\delta-\mathfrak{R}\rho)^2+(2t-\mathfrak{I}\rho)^2} = \mathfrak{R}\left(\frac{1}{1+\delta+2it-\rho}\right). \tag{4.5}$$

Note that, from Lemma 4.3.16, we have

$$\sum_{\rho\in\mathcal{Z}_{2t}} m_\zeta(\rho)\,\mathfrak{R}\left(\frac{1}{1+\delta+2it-\rho}\right) - \mathfrak{R}\left(\frac{\zeta'}{\zeta}(1+\delta+2it)\right) \le \left|\frac{\zeta'}{\zeta}(1+\delta+2it) - \sum_{\rho\in\mathcal{Z}_{2t}} \frac{m_\zeta(\rho)}{1+\delta+2it-\rho}\right| \ll \log|2t|.$$

Since $m_\zeta(\rho) \ge 0$ for all $\rho \in \mathcal{Z}_{2t}$, the inequality from Equation (4.5) tells us that by subtracting the sum from both sides we have

$$-\mathfrak{R}\left(\frac{\zeta'}{\zeta}(1+\delta+2it)\right) \ll \log|2t|.$$

Noting that $\log|2t| = \log(2) + \log|t| \le 2\log|t|$ completes the proof. □

Lemma 4.3.18 (ShiftOneBound)**.** There exists $C > 0$ such that for all $\delta \in (0,1)$ and $t \in \mathbb{R}$ with $|t| \ge 3$; if $\zeta(\rho) = 0$ with $\rho = \sigma + it$, then

$$-\mathfrak{R}\left(\frac{\zeta'}{\zeta}(1+\delta+it)\right) \le -\frac{1}{1+\delta-\sigma} + C\log|t|.$$

Proof. Note that for $\rho' \in \mathcal{Z}_t$

$$\Re\left(\frac{1}{1+\delta+it-\rho'}\right) = \Re\left(\frac{1+\delta-it-\overline{\rho'}}{(1+\delta+it-\rho')(1+\delta-it-\overline{\rho'})}\right)$$
$$= \frac{\Re(1+\delta-it-\overline{\rho'})}{|1+\delta+it-\rho'|^2} = \frac{1+\delta-\Re\rho'}{(1+\delta-\Re\rho')^2+(t-\Im\rho')^2}.$$

Now since $\rho' \in \mathcal{Z}_t$, we have that $|\rho-(3/2+it)| \le 5/6$. So, we have $\Re\rho' \in (2/3, 7/3)$ and $\Im\rho' \in (t-5/6, t+5/6)$. Thus we have that

$$1/3 < 1+\delta-\Re\rho' \qquad \text{and} \qquad (1+\delta-\Re\rho')^2+(t-\Im\rho')^2 < 16/9+25/36 = 89/36.$$

Which implies that

$$0 \le \frac{12}{89} < \frac{1+\delta-\Re\rho'}{(1+\delta-\Re\rho')^2+(t-\Im\rho')^2} = \Re\left(\frac{1}{1+\delta+it-\rho'}\right). \tag{4.6}$$

Note that, from Lemma 4.3.16, we have

$$\sum_{\rho\in\mathcal{Z}_t} m_\zeta(\rho)\,\Re\left(\frac{1}{1+\delta+it-\rho}\right) - \Re\left(\frac{\zeta'}{\zeta}(1+\delta+it)\right) \le \left|\frac{\zeta'}{\zeta}(1+\delta+it) - \sum_{\rho\in\mathcal{Z}_t}\frac{m_\zeta(\rho)}{1+\delta+it-\rho}\right| \ll \log|t|.$$

Since $m_\zeta(\rho) \ge 0$ for all $\rho' \in \mathcal{Z}_t$, the inequality from Equation (4.6) tells us that by subtracting the sum over all $\rho' \in \mathcal{Z}_t \setminus \{\rho\}$ from both sides we have

$$\frac{m_\zeta(\rho)}{\Re(1+\delta+it-\rho)} - \Re\left(\frac{\zeta'}{\zeta}(1+\delta+it)\right) \ll \log|t|.$$

But of course we have that $\Re(1+\delta+it-\rho) = 1+\delta-\sigma$. So subtracting this term from both sides and recalling the implied constant we have

$$-\Re\left(\frac{\zeta'}{\zeta}(1+\delta+it)\right) \le -\frac{m_\zeta(\rho)}{1+\delta-\sigma} + C\log|t|.$$

We have that $\sigma \le 1$ since ζ is zero free on the right half plane $\sigma > 1$. Thus $0 < 1+\delta-\sigma$. Noting this in combination with the fact that $1 \le m_\zeta(\rho)$ completes the proof. □

Lemma 4.3.19 (ShiftZeroBound)**.** For all $\delta \in (0,1)$ we have

$$-\Re\left(\frac{\zeta'}{\zeta}(1+\delta)\right) \le \frac{1}{\delta} + O(1).$$

Proof. From Theorem 3.4.14 we know that

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + O(1).$$

Changing variables $s \mapsto 1+\delta$ and applying the triangle inequality we have that

$$-\Re\left(\frac{\zeta'}{\zeta}(1+\delta)\right) \le \left|-\frac{\zeta'}{\zeta}(1+\delta)\right| \le \frac{1}{\delta} + O(1).$$

□

Lemma 4.3.20 (ThreeFourOneTrigIdentity)**.** We have that

$$0 \le 3 + 4\cos\theta + \cos 2\theta$$

for all $\theta \in \mathbb{R}$.

Proof. We know that $\cos(2\theta) = 2\cos^2\theta - 1$, thus

$$3 + 4\cos\theta + \cos 2\theta = 2 + 4\cos\theta + 2\cos^2\theta = 2\,(1 + \cos\theta)^2.$$

Noting that $0 \le 1 + \cos\theta$ completes the proof. ☐

Theorem 4.3.10 (ZeroInequality)**.** There exists a constant $0 < E < 1$ such that for all $\rho = \sigma + it$ with $\zeta(\rho) = 0$ and $|t| \ge 2$, one has

$$\sigma \le 1 - \frac{E}{\log|t|}.$$

Proof. From Theorem 3.5.1 when $\Re s > 1$ we have

$$-\frac{\zeta'}{\zeta}(s) = \sum_{1\le n} \frac{\Lambda(n)}{n^s}.$$

Thus,

$$-3\,\frac{\zeta'}{\zeta}(1+\delta) - 4\,\frac{\zeta'}{\zeta}(1+\delta+it) - \frac{\zeta'}{\zeta}(1+\delta+2it) = \sum_{1\le n} \Lambda(n)\, n^{-(1+\delta)}\left(3 + 4n^{-it} + n^{-2it}\right).$$

Now applying Euler's identity

$$\begin{aligned}-3\,\Re\left(\frac{\zeta'}{\zeta}(1+\delta)\right) - 4\,\Re\left(\frac{\zeta'}{\zeta}(1+\delta+it)\right) - \Re\left(\frac{\zeta'}{\zeta}(1+\delta+2it)\right)&\\ = \sum_{1\le n} \Lambda(n)\, n^{-(1+\delta)}\,(3 + 4\cos(-it\log n) + \cos(-2it\log n))&\end{aligned}$$

By Lemma 4.3.20 we know that the series on the right hand side is bounded below by 0, and by Lemmas 4.3.17, 4.3.18, and 4.3.19 we have an upper bound on the left hand side. So,

$$0 \le \frac{3}{\delta} + 3A - \frac{4}{1+\delta-\sigma} + 4B\log|t| + C\log|t|$$

where A, B, and C are the implied constants coming from Lemmas 4.3.19, 4.3.18, and 4.3.17 respectively. By choosing $D \ge 3A/\log 2 + 4B + C$ we have

$$\frac{4}{1+\delta-\sigma} \le \frac{3}{\delta} + D\log|t|$$

by some manipulation. Now if we choose $\delta = (2D\log|t|)^{-1}$ then we have

$$\frac{4}{1-\sigma+1/(2D\log|t|)} \le 7D\log|t|.$$

So with some manipulation we have that

$$\sigma \le 1 - \frac{1}{14D\log|t|}.$$

This is exactly the desired result with the constant $E = (14D)^{-1}$ ☐

Definition 4.3.9 (DeltaT)**.** Let $\delta_t = E/\log|t|$ where E is the constant coming from Theorem 4.3.10.

Lemma 4.3.21 (DeltaRange)**.** For all $t \in \mathbb{R}$ with $|t| \geq 2$ we have that

$$\delta_t < 1/14.$$

Proof. Note that $\delta_t = E/\log|t|$ where E is the implied constant from Lemma 4.3.10. But we know that $E = (14D)^{-1}$ where $D \geq 3A/\log 2 + 4B + C$ where A, B, and C are the constants coming from Lemmas 4.3.19, 4.3.18, and 4.3.17 respectively. Thus,

$$E \leq \frac{1}{14\,(3A/\log 2 + 4B + C)}.$$

But note that $A \geq 0$ and $B \geq 0$ by Lemmas 4.3.19 and 4.3.18 respectively. However, we have that

$$C \geq 2 + \frac{2\log((13\,\zeta(3/2))/(3\,\zeta(3)))}{\log 2}$$

by Theorem 4.3.9 with Lemmas 4.3.16 and 4.3.17. So, by a very lazy estimate we have $C \geq 2$ and $E \leq 1/28$. Thus,

$$\delta_t = \frac{E}{\log|t|} \leq \frac{1}{28\,\log 2} < \frac{1}{14}.$$

□

Lemma 4.3.22 (SumBoundII)**.** For all $t \in \mathbb{R}$ with $|t| \geq 2$ and $z = \sigma + it$ where $1 - \delta_t/3 \leq \sigma \leq 3/2$, we have that

$$\left|\frac{\zeta'}{\zeta}(z) - \sum_{\rho \in \mathcal{Z}_t} \frac{m_\zeta(\rho)}{z - \rho}\right| \ll \log|t|.$$

Proof. By Lemma 4.3.21 we have that

$$-11/21 < -1/2 - \delta_t/3 \leq \sigma - 3/2 \leq 0.$$

We apply Theorem 4.3.9 where $r' = 2/3$, $r = 3/4$, $R' = 5/6$, and $R = 8/9$. Thus for all $z \in \overline{\mathbb{D}_{5/6}} \setminus \mathcal{K}_f(5/6)$ we have that

$$\left|\frac{\zeta'}{\zeta}(z + 3/2 + it) - \sum_{\rho \in \mathcal{K}_f(5/6)} \frac{m_f(\rho)}{z - \rho}\right| \ll \log|t|$$

where $f(z) = \zeta(z + 3/2 + it)$ for $t \in \mathbb{R}$ with $|t| \geq 3$. Now if we let $z = \sigma - 3/2$, then $z \in (-11/21, 0) \subseteq \overline{\mathbb{D}_{5/6}}$. Additionally, $f(z) = \zeta(\sigma + it)$, where $\sigma + it$ lies in the zero free region given by Lemma 4.3.10 since $\sigma \geq 1 - \delta_t/3 \geq 1 - \delta_t$. Thus, $z \notin \mathcal{K}_f(5/6)$. So,

$$\left|\frac{\zeta'}{\zeta}(\sigma + it) - \sum_{\rho \in \mathcal{K}_f(5/6)} \frac{m_f(\rho)}{\sigma - 3/2 - \rho}\right| \ll \log|t|.$$

But now note that if $\rho \in \mathcal{K}_f(5/6)$, then $\zeta(\rho + 3/2 + it) = 0$ and $|\rho| \leq 5/6$. Additionally, note that $m_f(\rho) = m_\zeta(\rho + 3/2 + it)$. So changing variables using these facts gives us that

$$\left|\frac{\zeta'}{\zeta}(\sigma + it) - \sum_{\rho \in \mathcal{Z}_t} \frac{m_\zeta(\rho)}{\sigma + it - \rho}\right| \ll \log|t|.$$

□

Lemma 4.3.23 (GapSize)**.** Let $t \in \mathbb{R}$ with $|t| \geq 3$ and $z = \sigma + it$ where $1 - \delta_t/3 \leq \sigma \leq 3/2$. Additionally, let $\rho \in \mathcal{Z}_t$. Then we have that

$$|z - \rho| \geq \delta_t/6.$$

Proof. Let $\rho = \sigma' + it'$ and note that since $\rho \in \mathcal{Z}_t$, we have $t' \in (t - 5/6, t + 5/6)$. Thus, if $t > 1$ we have

$$\log|t'| \leq \log|t + 5/6| \leq \log|2t| = \log 2 + \log|t| \leq 2\log|t|.$$

And otherwise if $t < -1$ we have

$$\log|t'| \leq \log|t - 5/6| \leq \log|2t| = \log 2 + \log|t| \leq 2\log|t|.$$

So by taking reciprocals and multiplying through by a constant we have that $\delta_t \leq 2\delta_{t'}$. Now note that since $\rho \in \mathcal{Z}_t$ we know that $\sigma' \leq 1 - \delta_{t'}$ by Theorem 4.3.10 (here we use the fact that $|t| \geq 3$ to give us that $|t'| \geq 2$). Thus,

$$\delta_t/6 \leq \delta_{t'} - \delta_t/3 = 1 - \delta_t/3 - (1 - \delta_{t'}) \leq \sigma - \sigma' \leq |z - \rho|.$$

□

Lemma 4.3.24 (LogDerivZetaUniformLogSquaredBoundStrip)**.** There exists a constant $F \in (0, 1/2)$ such that for all $t \in \mathbb{R}$ with $|t| \geq 3$ one has

$$1 - \frac{F}{\log|t|} \leq \sigma \leq 3/2 \implies \left|\frac{\zeta'}{\zeta}(\sigma + it)\right| \ll \log^2|t|$$

where the implied constant is uniform in σ.

Proof. Take $F = E/3$ where E comes from Theorem 4.3.10. Then we have that $\sigma \geq 1 - \delta_t/3$. So, we apply Lemma 4.3.22, which gives us that

$$\left|\frac{\zeta'}{\zeta}(z) - \sum_{\rho \in \mathcal{Z}_t} \frac{m_\zeta(\rho)}{z - \rho}\right| \ll \log|t|.$$

Using the reverse triangle inequality and rearranging, we have that

$$\left|\frac{\zeta'}{\zeta}(z)\right| \leq \sum_{\rho \in \mathcal{Z}_t} \frac{m_\zeta(\rho)}{|z - \rho|} + C\,\log|t|$$

where C is the implied constant in Lemma 4.3.22. Now applying Lemma 4.3.23 we have that

$$\left|\frac{\zeta'}{\zeta}(z)\right| \leq \frac{6}{\delta_t} \sum_{\rho \in \mathcal{Z}_t} m_\zeta(\rho) + C\,\log|t|.$$

Now let $f(z) = \zeta(z + 3/2 + it)/\zeta(3/2 + it)$ with $\rho = \rho' + 3/2 + it$. Then if $\rho \in \mathcal{Z}_t$ we have that

$$0 = \zeta(\rho) = \zeta(\rho' + 3/2 + it) = f(\rho')$$

with the same multiplicity of zero, that is $m_\zeta(\rho) = m_f(\rho')$. And also if $\rho \in \mathcal{Z}_t$ then

$$5/6 \geq |\rho - (3/2 + it)| = |\rho'|.$$

Thus we change variables to have that

$$\left|\frac{\zeta'}{\zeta}(z)\right| \le \frac{6}{\delta_t} \sum_{\rho' \in \mathcal{K}_f(5/6)} m_f(\rho') + C \log|t|.$$

Now note that $f(0) = 1$ and for $|z| \le 8/9$ we have

$$|f(z)| = \frac{|\zeta(z + 3/2 + it)|}{|\zeta(3/2 + it)|} \le \frac{\zeta(3/2)}{\zeta(3)} \cdot (7 + 2|t|) \le \frac{13\,\zeta(3/2)}{3\,\zeta(3)}\,|t|$$

by Theorems 4.3.7 and 4.3.8. Thus by Theorem 4.3.4 we have that

$$\sum_{\rho' \in \mathcal{K}_f(5/6)} m_f(\rho') \le \frac{\log|t| + \log(13\,\zeta(3/2)/(3\,\zeta(3)))}{\log((8/9)/(5/6))} \le D \log|t|$$

where D is taken to be sufficiently large. Recall, by definition that, $\delta_t = E/\log|t|$ with E coming from Theorem 4.3.10. By using this fact and the above, we have that

$$\left|\frac{\zeta'}{\zeta}(z)\right| \ll \log^2|t| + \log|t|$$

where the implied constant is taken to be bigger than $\max(6D/E, C)$. We know that the RHS is bounded above by $\ll \log^2|t|$; so the result follows. □

Theorem 4.3.11 (LogDerivZetaUniformLogSquaredBound)**.** There exists a constant $F \in (0, 1/2)$ such that for all $t \in \mathbb{R}$ with $|t| \ge 3$ one has

$$1 - \frac{F}{\log|t|} \le \sigma \implies \left|\frac{\zeta'}{\zeta}(\sigma + it)\right| \ll \log^2|t|$$

where the implied constant is uniform in σ.

Proof. Note that

$$\left|\frac{\zeta'}{\zeta}(\sigma + it)\right| = \sum_{1 \le n} \frac{\Lambda(n)}{|n^{\sigma+it}|} = \sum_{1 \le n} \frac{\Lambda(n)}{n^\sigma} = -\frac{\zeta'}{\zeta}(\sigma) \le \left|\frac{\zeta'}{\zeta}(\sigma)\right|.$$

From Theorem 3.4.14, and applying the triangle inequality we know that

$$\left|\frac{\zeta'}{\zeta}(s)\right| \le \frac{1}{|s-1|} + C.$$

where $C > 0$ is some constant. Thus, for $\sigma \ge 3/2$ we have that

$$\left|\frac{\zeta'}{\zeta}(\sigma + it)\right| \le \left|\frac{\zeta'}{\zeta}(\sigma)\right| \le \frac{1}{\sigma - 1} + C \le 2 + C \ll 1 \ll \log^2|t|.$$

Putting this together with Lemma 4.3.24 completes the proof. □

Theorem 4.3.12 (LogDerivZetaLogSquaredBoundSmallt)**.** For $T > 0$ and $\sigma' = 1 - \delta_T/3 = 1 - F/\log T$, if $|t| \le T$ then we have that

$$\left|\frac{\zeta'}{\zeta}(\sigma' + it)\right| \ll \log^2(2 + T).$$

Proof. Note that if $|t| \geq 3$ then from Theorem 4.3.11 we have that

$$\left|\frac{\zeta'}{\zeta}(\sigma' + it)\right| \ll \log^2 |t| \leq \log^2 T \leq \log^2(2+T).$$

Otherwise, if $|t| \leq 3$, then from Theorem 3.4.14 and applying the triangle inequality we know

$$\left|\frac{\zeta'}{\zeta}(\sigma' + it)\right| \leq \frac{1}{|(\sigma' - 1) + it|} + C \leq \frac{\log T}{F} + C$$

where $C \geq 0$. Thus, we have that

$$\left|\frac{\zeta'}{\zeta}(\sigma' + it)\right| \leq \left(\frac{\log T}{F \log 2} + \frac{C}{\log 2}\right) \log(2+|t|) \leq \left(\frac{\log(2+T)}{F \log 2} + \frac{C}{\log 2}\right) \log(2+T) \ll \log^2(2+T).$$

□

From here out we closely follow our previous proof of the Medium PNT and we modify it using our new estimate in Theorem 4.3.11. Recall Definition 3.5.2; for fixed $\varepsilon > 0$ and a bump function ν supported on $[1/2, 2]$ we have

$$\psi_\varepsilon(X) = \frac{1}{2\pi i} \int_{(\sigma)} \left(-\frac{\zeta'}{\zeta}(s)\right) \mathcal{M}(\tilde{1}_\varepsilon)(s)\, X^s\, ds$$

where $\sigma = 1 + 1/\log X$. Let $T > 3$ be a large constant to be chosen later, and we take $\sigma' = 1 - \delta_T/3 = 1 - F/\log T$ with F coming from Theorem 4.3.11. We integrate along the σ vertical line, and we pull contours accumulating the pole at $s = 1$ when we integrate along the curves

- I_1: $\sigma - i\infty$ to $\sigma - iT$
- I_2: $\sigma' - iT$ to $\sigma - iT$
- I_3: $\sigma' - iT$ to $\sigma' + iT$
- I_4: $\sigma' + iT$ to $\sigma + iT$
- I_5: $\sigma + iT$ to $\sigma + i\infty$.

Definition 4.3.10 (I1New)**.** Let

$$I_1(\nu, \varepsilon, X, T) = \frac{1}{2\pi i} \int_{-\infty}^{-T} \left(-\frac{\zeta'}{\zeta}(\sigma + it)\right) \mathcal{M}(\tilde{1}_\varepsilon)(\sigma + it)\, X^{\sigma + it}\, dt.$$

Definition 4.3.11 (I5New)**.** Let

$$I_5(\nu, \varepsilon, X, T) = \frac{1}{2\pi i} \int_{T}^{\infty} \left(-\frac{\zeta'}{\zeta}(\sigma + it)\right) \mathcal{M}(\tilde{1}_\varepsilon)(\sigma + it)\, X^{\sigma + it}\, dt.$$

Lemma 4.3.25 (I1NewBound)**.** We have that

$$|I_1(\nu, \varepsilon, X, T)| \ll \frac{X}{\varepsilon\sqrt{T}}.$$

Proof. Note that $|I_1(\nu,\varepsilon,X,T)| =$

$$\left|\frac{1}{2\pi i}\int_{-\infty}^{-T}\left(-\frac{\zeta'}{\zeta}(\sigma+it)\right)\mathcal{M}(\tilde{1}_\varepsilon)(\sigma+it)\,X^{\sigma+it}\,dt\right| \ll \int_{-\infty}^{-T}\left|\frac{\zeta'}{\zeta}(\sigma+it)\right|\cdot|\mathcal{M}(\tilde{1}_\varepsilon)(\sigma+it)|\cdot X^{\sigma}\,dt.$$

Applying Theorem 4.3.11 and Lemma 3.3.10, we have that

$$|I_1(\nu,\varepsilon,X,T)| \ll \int_{-\infty}^{-T}\log^2|t|\cdot\frac{X^\sigma}{\varepsilon\,|\sigma+it|^2}\,dt \ll \frac{X}{\varepsilon}\int_T^\infty\frac{\sqrt{t}\,dt}{t^2} \ll \frac{X}{\varepsilon\sqrt{T}}.$$

Here we are using the fact that $\log^2 t$ grows slower than $\sqrt{t}$, $|\sigma+it|^2 \geq t^2$, and $X^\sigma = X\cdot X^{1/\log X} = eX$. □

Lemma 4.3.26 (I5NewBound)**.** We have that

$$|I_5(\nu,\varepsilon,X,T)| \ll \frac{X}{\varepsilon\sqrt{T}}.$$

Proof. By symmetry, note that

$$|I_1(\nu,\varepsilon,X,T)| = |\overline{I_5(\nu,\varepsilon,X,T)}| = |I_5(\nu,\varepsilon,X,T)|.$$

Applying Lemma 4.3.25 completes the proof. □

Definition 4.3.12 (I2New)**.** Let

$$I_2(\nu,\varepsilon,X,T) = \frac{1}{2\pi i}\int_{\sigma'}^{\sigma}\left(-\frac{\zeta'}{\zeta}(\sigma_0-iT)\right)\mathcal{M}(\tilde{1}_\varepsilon)(\sigma_0-iT)\,X^{\sigma_0-iT}\,d\sigma_0.$$

Definition 4.3.13 (I4New)**.** Let

$$I_4(\nu,\varepsilon,X,T) = \frac{1}{2\pi i}\int_{\sigma'}^{\sigma}\left(-\frac{\zeta'}{\zeta}(\sigma_0+iT)\right)\mathcal{M}(\tilde{1}_\varepsilon)(\sigma_0+iT)\,X^{\sigma_0+iT}\,d\sigma_0.$$

Lemma 4.3.27 (I2NewBound)**.** We have that

$$|I_2(\nu,\varepsilon,X,T)| \ll \frac{X}{\varepsilon\sqrt{T}}.$$

Proof. Note that $|I_2(\nu,\varepsilon,X,T)| =$

$$\left|\frac{1}{2\pi i}\int_{\sigma'}^{\sigma}\left(-\frac{\zeta'}{\zeta}(\sigma_0-iT)\right)\mathcal{M}(\tilde{1}_\varepsilon)(\sigma_0-iT)\,X^{\sigma_0-iT}\,d\sigma_0\right| \ll \int_{\sigma'}^{\sigma}\left|\frac{\zeta'}{\zeta}(\sigma_0-iT)\right|\cdot|\mathcal{M}(\tilde{1}_\varepsilon)(\sigma_0-iT)|\cdot X^{\sigma_0}\,d\sigma_0.$$

Applying Theorem 4.3.11 and Lemma 3.3.10, we have that

$$|I_2(\nu,\varepsilon,X,T)| \ll \int_{\sigma'}^{\sigma}\log^2 T\cdot\frac{X^{\sigma_0}}{\varepsilon\,|\sigma_0-iT|^2}\,d\sigma_0 \ll \frac{X\,\log^2 T}{\varepsilon\,T^2}\int_{\sigma'}^{\sigma}d\,\sigma_0 = \frac{X\,\log^2 T}{\varepsilon\,T^2}\,(\sigma-\sigma').$$

Here we are using the fact that $X^{\sigma_0} \leq X^\sigma = X\cdot X^{1/\log X} = eX$ and $|\sigma_0 - iT|^2 \geq T^2$. Now note that

$$|I_2(\nu,\varepsilon,X,T)| \ll \frac{X\,\log^2 T}{\varepsilon\,T^2}\,(\sigma-\sigma') = \frac{X\,\log^2 T}{\varepsilon\,T^2\,\log X} + \frac{FX\,\log T}{\varepsilon\,T^2} \ll \frac{X}{\varepsilon\sqrt{T}}.$$

Here we are using the fact that $\log T \ll T^{3/2}$, $\log^2 T \ll T^{3/2}$, and $X/\log X \ll X$. □

Lemma 4.3.28 (I4NewBound)**.** We have that

$$|I_4(\nu, \varepsilon, X, T)| \ll \frac{X}{\varepsilon\sqrt{T}}.$$

Proof. By symmetry, note that

$$|I_2(\nu, \varepsilon, X, T)| = |\overline{I_4(\nu, \varepsilon, X, T)}| = |I_4(\nu, \varepsilon, X, T)|.$$

Applying Lemma 4.3.27 completes the proof. □

Definition 4.3.14 (I3New)**.** Let

$$I_3(\nu, \varepsilon, X, T) = \frac{1}{2\pi i} \int_{-T}^{T} \left(-\frac{\zeta'}{\zeta}(\sigma' + it) \right) \mathcal{M}(\tilde{1}_\varepsilon)(\sigma' + it)\, X^{\sigma' + it}\, dt.$$

Lemma 4.3.29 (I3NewBound)**.** We have that

$$|I_3(\nu, \varepsilon, X, T)| \ll \frac{X^{1-F/\log T}\sqrt{T}}{\varepsilon}.$$

Proof. Note that $|I_3(\nu, \varepsilon, X, T)| =$

$$\left| \frac{1}{2\pi i} \int_{-T}^{T} \left(-\frac{\zeta'}{\zeta}(\sigma' + it) \right) \mathcal{M}(\tilde{1}_\varepsilon)(\sigma' + it)\, X^{\sigma' + it}\, dt \right| \ll \int_{-T}^{T} \left| \frac{\zeta'}{\zeta}(\sigma' + it) \right| \cdot |\mathcal{M}(\tilde{1}_\varepsilon)(\sigma' + it)| \cdot X^{\sigma'}\, dt.$$

Applying Theorem 4.3.12 and Lemma 3.3.10, we have that

$$|I_3(\nu, \varepsilon, X, T)| \ll \int_{-T}^{T} \log^2(2 + T) \cdot \frac{X^{\sigma'}}{\varepsilon\, |\sigma' + it|^2}\, dt \ll \frac{X^{1-F/\log T}\sqrt{T}}{\varepsilon} \int_{0}^{T} \frac{dt}{|\sigma' + it|^2}.$$

Here we are using the fact that this integrand is symmetric in t about 0 and that $\log^2(2 + T) \ll \sqrt{T}$ for sufficiently large T. Now note that, by Lemma 4.3.21, we have

$$\frac{1}{|\sigma' + it|^2} = \frac{1}{(1 - \delta_T/3)^2 + t^2} < \frac{1}{(41/42)^2 + t^2}.$$

Thus,

$$|I_3(\nu, \varepsilon, X, T)| \ll \frac{X^{1-F/\log T}\sqrt{T}}{\varepsilon} \int_{0}^{T} \frac{dt}{|\sigma' + it|^2} \leq \frac{X^{1-F/\log T}\sqrt{T}}{\varepsilon} \int_{0}^{\infty} \frac{dt}{(41/42)^2 + t^2}.$$

The integral on the right hand side evaluates to $21\pi/41$, which is just a constant, so the desired result follows. □

Theorem 4.3.13 (SmoothedChebyshevPull3)**.** We have that

$$\psi_\varepsilon(X) = \mathcal{M}(\tilde{1}_\varepsilon)(1)\, X^1 + I_1 - I_2 + I_3 + I_4 + I_5.$$

Proof. Pull contours and accumulate the pole of ζ'/ζ at $s = 1$. □

Theorem 4.3.14 (StrongPNT)**.** We have

$$\sum_{n \leq x} \Lambda(n) = x + O\left(x \exp(-c\sqrt{\log x}) \right).$$

Proof. By Theorem 3.5.3 and 4.3.13 we have that

$$\mathcal{M}(\widetilde{1}_\varepsilon)(1)\,x^1 + I_1 - I_2 + I_3 + I_4 + I_5 = \psi(x) + O(\varepsilon x \log x).$$

Applying Theorem 3.3.11 and Lemmas 4.3.25, 4.3.27, 4.3.29, 4.3.28, and 4.3.26 we have that

$$\psi(x) = x + O(\varepsilon x) + O(\varepsilon x \log x) + O\left(\frac{x}{\varepsilon\sqrt{T}}\right) + O\left(\frac{x^{1-F/\log T}\sqrt{T}}{\varepsilon}\right).$$

We absorb the $O(\varepsilon x)$ term into the $O(\varepsilon x \log x)$ term and balance the last two terms in T.

$$\frac{x}{\varepsilon\sqrt{T}} = \frac{x^{1-F/\log T}\sqrt{T}}{\varepsilon} \implies T = \exp(\sqrt{F \log x}).$$

Thus,

$$\psi(x) = x + O(\varepsilon x \log x) + O\left(\frac{x}{\varepsilon \exp((1/2)\cdot\sqrt{F\log x})}\right).$$

Now we balance the last two terms in ε.

$$\varepsilon x \log x = \frac{x}{\varepsilon \exp((1/2)\cdot\sqrt{F\log x})} \implies \varepsilon \log x = \frac{\sqrt{\log x}}{\exp((1/4)\cdot\sqrt{F\log x})}.$$

Thus,

$$\psi(x) = x + O\left(x \exp(-(\sqrt{F}/4)\cdot\sqrt{\log x})\sqrt{\log x}\right).$$

Absorbing the $\sqrt{\log x}$ into the $\exp(-(\sqrt{F}/4)\cdot\sqrt{\log x})$ completes the proof. □

Chapter 5

Elementary Corollaries

Lemma 5.0.1 (finsum-range-eq-sum-range)**.** For any arithmetic function f and real number x, one has

$$\sum_{n \leq x} f(n) = \sum_{n \leq \lfloor x \rfloor_+} f(n)$$

and

$$\sum_{n < x} f(n) = \sum_{n < \lceil x \rceil_+} f(n).$$

Proof. Straightforward. □

Theorem 5.0.1 (chebyshev-asymptotic)**.** One has

$$\sum_{p \leq x} \log p = x + o(x).$$

Proof. From the prime number theorem we already have

$$\sum_{n \leq x} \Lambda(n) = x + o(x)$$

so it suffices to show that

$$\sum_{j \geq 2} \sum_{p^j \leq x} \log p = o(x).$$

Only the terms with $j \leq \log x / \log 2$ contribute, and each j contributes at most $\sqrt{x} \log x$ to the sum, so the left-hand side is $O(\sqrt{x} \log^2 x) = o(x)$ as required. □

Corollary 5.0.1 (primorial-bounds)**.** We have

$$\prod_{p \leq x} p = \exp(x + o(x))$$

Proof. Exponentiate Theorem **??**. □

Theorem 5.0.2 (pi-asymp)**.** There exists a function $c(x)$ such that $c(x) = o(1)$ as $x \to \infty$ and

$$\pi(x) = (1 + c(x)) \int_2^x \frac{dt}{\log t}$$

for all x large enough.

Proof. We have the identity

$$\pi(x) = \frac{1}{\log x} \sum_{p \le x} \log p + \int_2^x (\sum_{p \le t} \log p) \frac{dt}{t \log^2 t}$$

as can be proven by interchanging the sum and integral and using the fundamental theorem of calculus. For any ε, we know from Theorem **??** that there is x_ε such that $\sum_{p \le t} \log p = t + O(\varepsilon t)$ for $t \ge x_\varepsilon$, hence for $x \ge x_\varepsilon$

$$\pi(x) = \frac{1}{\log x}(x + O(\varepsilon x)) + \int_{x_\varepsilon}^x (t + O(\varepsilon t)) \frac{dt}{t \log^2 t} + O_\varepsilon(1)$$

where the $O_\varepsilon(1)$ term can depend on x_ε but is independent of x. One can evaluate this after an integration by parts as

$$\pi(x) = (1 + O(\varepsilon)) \int_{x_\varepsilon}^x \frac{dt}{\log t} + O_\varepsilon(1)$$

$$= (1 + O(\varepsilon)) \int_2^x \frac{dt}{\log t}$$

for x large enough, giving the claim. □

Corollary 5.0.2 (pi-alt)**.** One has

$$\pi(x) = (1 + o(1)) \frac{x}{\log x}$$

as $x \to \infty$.

Proof. An integration by parts gives

$$\int_2^x \frac{dt}{\log t} = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{dt}{\log^2 t}.$$

We have the crude bounds

$$\int_2^{\sqrt{x}} \frac{dt}{\log^2 t} = O(\sqrt{x})$$

and

$$\int_{\sqrt{x}}^x \frac{dt}{\log^2 t} = O(\frac{x}{\log^2 x})$$

and combining all this we obtain

$$\int_2^x \frac{dt}{\log t} = \frac{x}{\log x} + O(\frac{x}{\log^2 x})$$

$$= (1 + o(1)) \frac{x}{\log x}$$

and the claim then follows from Theorem 5.0.2. □

Let p_n denote the n^{th} prime.

Proposition 5.0.1 (pn-asymptotic)**.** One has

$$p_n = (1 + o(1))n \log n$$

as $n \to \infty$.

Proof. Use Corollary 5.0.2 to show that $n = \pi(p_n) \sim p_n/\log p_n$ Taking logs gives $\log n \sim \log p_n - \log\log p_n \sim \log p_n$. Multiplying these gives $p_n \sim n \log n$ from which the result follows. □

Corollary 5.0.3 (pn-pn-plus-one)**.** We have $p_{n+1} - p_n = o(p_n)$ as $n \to \infty$.

Proof. Easy consequence of preceding proposition. □

Corollary 5.0.4 (prime-between)**.** For every $\varepsilon > 0$, there is a prime between x and $(1+\varepsilon)x$ for all sufficiently large x.

Proof. Use Corollary 5.0.2 to show that $\pi((1 + \varepsilon)x) - \pi(x)$ goes to infinity as $x \to \infty$. □

Proposition 5.0.2. We have $|\sum_{n\le x} \frac{\mu(n)}{n}| \le 1$.

Proof. From Möbius inversion $1_{n=1} = \sum_{d|n} \mu(d)$ and summing we have

$$1 = \sum_{d\le x} \mu(d) \lfloor \frac{x}{d} \rfloor$$

for any $x \ge 1$. Since $\lfloor \frac{x}{d} \rfloor = \frac{x}{d} - \epsilon_d$ with $0 \le \epsilon_d < 1$ and $\epsilon_x = 0$, we conclude that

$$1 \ge x \sum_{d\le x} \frac{\mu(d)}{d} - (x - 1)$$

and the claim follows. □

Proposition 5.0.3 (Möbius form of prime number theorem)**.** We have $\sum_{n\le x} \mu(n) = o(x)$.

Proof. From the Dirichlet convolution identity

$$\mu(n) \log n = -\sum_{d|n} \mu(d)\Lambda(n/d)$$

and summing we obtain

$$\sum_{n\le x} \mu(n) \log n = -\sum_{d\le x} \mu(d) \sum_{m\le x/d} \Lambda(m).$$

For any $\varepsilon > 0$, we have from the prime number theorem that

$$\sum_{m\le x/d} \Lambda(m) = x/d + O(\varepsilon x/d) + O_\varepsilon(1)$$

(divide into cases depending on whether x/d is large or small compared to ε). We conclude that

$$\sum_{n\le x} \mu(n) \log n = -x \sum_{d\le x} \frac{\mu(d)}{d} + O(\varepsilon x \log x) + O_\varepsilon(x).$$

Applying (5.0.2) we conclude that

$$\sum_{n\leq x}\mu(n)\log n = O(\varepsilon x\log x) + O_\varepsilon(x).$$

and hence

$$\sum_{n\leq x}\mu(n)\log x = O(\varepsilon x\log x) + O_\varepsilon(x) + O(\sum_{n\leq x}(\log x - \log n)).$$

From Stirling's formula one has

$$\sum_{n\leq x}(\log x - \log n) = O(x)$$

thus

$$\sum_{n\leq x}\mu(n)\log x = O(\varepsilon x\log x) + O_\varepsilon(x)$$

and thus

$$\sum_{n\leq x}\mu(n) = O(\varepsilon x) + O_\varepsilon(\frac{x}{\log x}).$$

Sending $\varepsilon \to 0$ we obtain the claim. □

Proposition 5.0.4. We have $\sum_{n\leq x}\lambda(n) = o(x)$.

Proof. From the identity

$$\lambda(n) = \sum_{d^2|n}\mu(n/d^2)$$

and summing, we have

$$\sum_{n\leq x}\lambda(n) = \sum_{d\leq\sqrt{x}}\sum_{n\leq x/d^2}\mu(n).$$

For any $\varepsilon > 0$, we have from Proposition 5.0.3 that

$$\sum_{n\leq x/d^2}\mu(n) = O(\varepsilon x/d^2) + O_\varepsilon(1)$$

and hence on summing in d

$$\sum_{n\leq x}\lambda(n) = O(\varepsilon x) + O_\varepsilon(x^{1/2}).$$

Sending $\varepsilon \to 0$ we obtain the claim. □

Proposition 5.0.5 (Alternate Möbius form of prime number theorem)**.** We have $\sum_{n\leq x}\mu(n)/n = o(1)$.

Proof. As in the proof of Theorem 5.0.2, we have

$$1 = \sum_{d\leq x}\mu(d)\lfloor\frac{x}{d}\rfloor$$

$$= x\sum_{d\leq x}\frac{\mu(d)}{d} - \sum_{d\leq x}\mu(d)\{\frac{x}{d}\}$$

so it will suffice to show that

$$\sum_{d\leq x}\mu(d)\{\frac{x}{d}\}=o(x).$$

Let N be a natural number. It suffices to show that

$$\sum_{d\leq x}\mu(d)\{\frac{x}{d}\}=O(x/N).$$

if x is large enough depending on N. We can split the left-hand side as the sum of

$$\sum_{d\leq x/N}\mu(d)\{\frac{x}{d}\}$$

and

$$\sum_{j=1}^{N-1}\sum_{x/(j+1)<d\leq x/j}\mu(d)(x/d-j).$$

The first term is clearly $O(x/N)$. For the second term, we can use Theorem 5.0.3 and summation by parts (using the fact that $x/d-j$ is monotone and bounded) to find that

$$\sum_{x/(j+1)<d\leq x/j}\mu(d)(x/d-j)=o(x)$$

for any given j, so in particular

$$\sum_{x/(j+1)<d\leq x/j}\mu(d)(x/d-j)=O(x/N^2)$$

for all $j=1,\dots,N-1$ if x is large enough depending on N. Summing all the bounds, we obtain the claim. □

5.1 Consequences of the PNT in arithmetic progressions

Theorem 5.1.1 (Prime number theorem in AP)**.** If $a\ (q)$ is a primitive residue class, then one has

$$\sum_{p\leq x:p=a\ (q)}\log p=\frac{x}{\phi(q)}+o(x).$$

Proof. This is a routine modification of the proof of Theorem **??**. □

Corollary 5.1.1 (Dirichlet's theorem)**.** Any primitive residue class contains an infinite number of primes.

Proof. If this were not the case, then the sum $\sum_{p\leq x:p=a\ (q)}\log p$ would be bounded in x, contradicting Theorem 5.1.1. □

5.2 Consequences of the Chebotarev density theorem

Lemma 5.2.1 (Cyclotomic Chebotarev)**.** For any a coprime to m,

$$\sum_{N\mathfrak{p}\leq x;N\mathfrak{p}=a\ (m)}\log N\mathfrak{p}=\frac{1}{|G|}\sum_{N\mathfrak{p}\leq x}\log N\mathfrak{p}.$$

Proof. This should follow from Lemma 2.7.1 by a Fourier expansion. □

Chapter 6

Explicit estimates

We will try to systematically collect explicit estimates related to the prime number theorem from the literature, and formalize them in a modular fashion. We divide such estimates into four classes:

- *Zeta function* explicit estimates: bounds on the zeta function and its zeroes.
- *Primary* explicit estimates: those that are directly control $\psi(x)$ and $M(x)$, usually via information on the zeta function.
- *Secondary* explicit estimates: these are useful general-purpose estimates on functions relating to the primes, such as bounds on the n-th prime, or estimates for the prime counting function $\pi(x)$. These are generally derived from primary estimates and elementary arguments.
- *Tertiary* explicit estimates: these are bespoke applications to particular problems in analytic number theory or combinatorics that often require secondary estimates as input.

In this project we will state the best available zeta and primary estimates known in the literature, and try to formalize at least some of them; state the best available secondary estimates known in the literature, as well as various tools from passing from primary to secondary estimates, and formalize these; and then finally formalize some tertiary estimates as applications of the secondary ones.

Chapter 7

Zeta function estimates

7.1 Definitions

Definition 7.1.1. ρ is understood to lie in the set $\{s : \zeta(s) = 0\}$, counted with multiplicity. We will often restrict the zeroes ρ to a rectangle $\{\Re\rho \in I, \Im\rho \in J\}$, for instance through sums of the form $\sum_{\Re\rho \in I, \Im\rho \in J} f(\rho)$.

Definition 7.1.2. We say that the Riemann hypothesis has been verified up to height T if there are no zeroes in the rectangle $\{\Re\rho \in (0.5, 1), \Im\rho \in [0, T]\}$.

Definition 7.1.3 (Section 1.1, FKS2)**.** We say that one has a classical zero-free region with parameter R if $zeta(s)$ has no zeroes in the region $Re(s) \geq 1 - 1/R * \log|\Im s|$ for $\Im(s) > 3$.

Definition 7.1.4 (Zero counting function N(T))**.** The number of zeroes of imaginary part between 0 and T, counting multiplicity

Definition 7.1.5 (Riemann von Mangoldt estimate)**.** An estimate of the form $N(T) - \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8}| \leq b_1 \log T + b_2 \log\log T + b_3$ for $T \geq 2$.

Definition 7.1.6 (Zero density bound)**.** An estimate of the form $N(\sigma, T) \leq c_1 T^p \log^q T + c_2 \log^2 T - \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8}| \leq b_1 \log T + b_2 \log\log T + b_3$ for $T \geq 2$.

7.2 The estimates of Kadiri, Lumley, and Ng

In this section we establish the primary results of [9].

7.3 The zeta function bounds of Rosser and Schoenfeld

In this section we formalize the zeta function bounds of Rosser and Schoenfeld.

Theorem 7.3.1 (Rosser–Schoenfeld Theorem 19)**.** One has a Riemann von Mangoldt estimate with parameters 0.137, 0.443, and 1.588.

Proof. □

7.4 Approximating the Riemann zeta function

We want a good explicit estimate on

$$\sum_{n\leq a} \frac{1}{n^s} - \int_0^a \frac{du}{u^s},$$

for a a half-integer. As it turns out, this is the same problem as that of approximating $\zeta(s)$ by a sum $\sum_{n\leq a} n^{-s}$. This is one of the two[1] main, standard ways of approximating $\zeta(s)$.

The non-explicit version of the result was first proved in [8, Lemmas 1 and 2]. The proof there uses first-order Euler-Maclaurin combined with a decomposition of $\lfloor x\rfloor - x + 1/2$ that turns out to be equivalent to Poisson summation. The exposition in [14, §4.7–4.11] uses first-order Euler-Maclaurin and van de Corput's Process B; the main idea of the latter is Poisson summation.

There are already several explicit versions of the result in the literature. In [2], [**?**] and [12], what we have is successively sharper explicit versions of Hardy and Littlewood's original proof. The proof in [4, Lemma 2.10] proceeds simply by a careful estimation of the terms in high-order Euler-Maclaurin; it does not use Poisson summation. Finally, [3] is an explicit version of [14, §4.7–4.11]; it gives a weaker bound than [12] or [4]. The strongest of these results is [12].

We will give another version here, in part because we wish to relax conditions – we will work with $|\Im s| < 2\pi a$ rather than $|\Im s| \leq a$ – and in part to show that one can prove an asymptotically optimal result easily and concisely. We will use first-order Euler-Maclaurin and Poisson summation. We assume that a is a half-integer; if one inserts the same assumption into [4, Lemma 2.10], one can improve the result there, yielding an error term closer to the one here.

For additional context, see the Zulip discussion at `https://leanprover.zulipchat.com/#narrow/channel/423402-PrimeNumberTheorem.2B/topic/Let.20us.20formalize.20an.20appendix`

Definition 7.4.1 (e)**.** We recall that $e(\alpha) = e^{2\pi i\alpha}$.

7.4.1 The decay of a Fourier transform

Our first objective will be to estimate the Fourier transform of $t^{-s}\mathbb{1}_{[a,b]}$. In particular, we will show that, if a and b are half-integers, the Fourier cosine transform has quadratic decay *when evaluated at integers.* In general, for real arguments, the Fourier transform of a discontinuous function such as $t^{-s}\mathbb{1}_{[a,b]}$ does not have quadratic decay.

Lemma 7.4.1 (Fourier transform of a truncated power law)**.** Let $s = \sigma + i\tau$, $\sigma \geq 0$, $\tau \in \mathbb{R}$. Let $\nu \in \mathbb{R} \setminus \{0\}$, $b > a > \frac{|\tau|}{2\pi|\nu|}$. Then

$$\int_a^b t^{-s} e(\nu t) dt = \left.\frac{t^{-\sigma} e(\varphi_\nu(t))}{2\pi i \varphi_\nu'(t)}\right|_a^b + \sigma \int_a^b \frac{t^{-\sigma-1}}{2\pi i \varphi_\nu'(t)} e(\varphi_\nu(t)) dt + \int_a^b \frac{t^{-\sigma}\varphi_\nu''(t)}{2\pi i (\varphi_\nu'(t))^2} e(\varphi_\nu(t)) dt, \tag{7.1}$$

where $\varphi_\nu(t) = \nu t - \frac{\tau}{2\pi}\log t$.

[1]The other one is the approximate functional equation.

Proof. We write $t^{-s}e(\nu t) = t^{-\sigma}e(\varphi_\nu(t))$ and integrate by parts with $u = t^{-\sigma}/(2\pi i\varphi'_\nu(t))$, $v = e(\varphi_\nu(t))$. Here $\varphi'_\nu(t) = \nu - \tau/(2\pi t) \neq 0$ for $t \in [a,b]$ because $t \geq a > |\tau|/(2\pi|\nu|)$ implies $|\nu| > |\tau|/(2\pi t)$. Clearly

$$udv = \frac{t^{-\sigma}}{2\pi i\varphi'_\nu(t)} \cdot 2\pi i\varphi'_\nu(t)e(\varphi_\nu(t))dt = t^{-\sigma}e(\varphi_\nu(t))dt,$$

while

$$du = \left(\frac{-\sigma t^{-\sigma-1}}{2\pi i\varphi'_\nu(t)} - \frac{t^{-\sigma}\varphi''_\nu(t)}{2\pi i(\varphi'_\nu(t))^2}\right) dt.$$

□

Lemma 7.4.2 (Total variation of a function with monotone absolute value)**.** Let $g : [a,b] \to \mathbb{R}$ be continuous, with $|g(t)|$ non-increasing. Then g is monotone, and $\|g\|_{\mathrm{TV}} = |g(a)| - |g(b)|$.

Proof. Suppose g changed sign: $g(a') > 0 > g(b')$ or $g(a') < 0 < g(b')$ for some $a \leq a' < b' \leq b$. By IVT, there would be an $r \in [a', b']$ such that $g(r) = 0$. Since $|g|$ is non-increasing, $g(b') = 0$; contradiction. So, g does not change sign: either $g \leq 0$ or $g \geq 0$.

Thus, there is an $\varepsilon \in \{-1, 1\}$ such that $g(t) = \varepsilon|g(t)|$ for all $t \in [a,b]$. Hence, g is monotone. Then $\|g\|_{\mathrm{TV}} = |g(a) - g(b)|$. Since $|g(a)| \geq |g(b)|$ and $g(a)$, $g(b)$ are either both non-positive or non-negative, $|g(a) - g(b)| = |g(a)| - |g(b)|$. □

Lemma 7.4.3 (Non-stationary phase estimate)**.** Let $\varphi : [a,b] \to \mathbb{R}$ be C^1 with $\varphi'(t) \neq 0$ for all $t \in [a,b]$. Let $h : [a,b] \to \mathbb{R}$ be such that $g(t) = h(t)/\varphi'(t)$ is continuous and $|g(t)|$ is non-increasing. Then

$$\left|\int_a^b h(t)e(\varphi(t))dt\right| \leq \frac{|g(a)|}{\pi}.$$

Proof. Since φ is C^1, $e(\varphi(t))$ is C^1, and $h(t)e(\varphi(t)) = \frac{h(t)}{2\pi i\varphi'(t)}\frac{d}{dt}e(\varphi(t))$ everywhere. By Lemma 7.4.2, g is of bounded variation. Hence, we can integrate by parts:

$$\int_a^b h(t)e(\varphi(t))dt = \left.\frac{h(t)e(\varphi(t))}{2\pi i\varphi'(t)}\right|_a^b - \int_a^b e(\varphi(t))\, d\left(\frac{h(t)}{2\pi i\varphi'(t)}\right).$$

The first term on the right has absolute value $\leq \frac{|g(a)|+|g(b)|}{2\pi}$. Again by Lemma 7.4.2,

$$\left|\int_a^b e(\varphi(t))\, d\left(\frac{h(t)}{2\pi i\varphi'(t)}\right)\right| \leq \frac{1}{2\pi}\|g\|_{\mathrm{TV}} = \frac{|g(a)| - |g(b)|}{2\pi}.$$

We are done by $\frac{|g(a)|+|g(b)|}{2\pi} + \frac{|g(a)|-|g(b)|}{2\pi} = \frac{|g(a)|}{\pi}$. □

Lemma 7.4.4 (A decreasing function)**.** Let $\sigma \geq 0$, $\tau \in \mathbb{R}$, $\nu \in \mathbb{R} \setminus \{0\}$. Let $b > a > \frac{|\tau|}{2\pi|\nu|}$. Then, for any $k \geq 1$, $f(t) = t^{-\sigma-k}|2\pi\nu - \tau/t|^{-k-1}$ is decreasing on $[a,b]$.

Proof. Let $a \leq t \leq b$. Since $\left|\frac{\tau}{t\nu}\right| < 2\pi$, we see that $2\pi - \frac{\tau}{\nu t} > 0$, and so $|2\pi\nu - \tau/t|^{-k-1} = |\nu|^{-k-1}\left(2\pi - \frac{\tau}{t\nu}\right)^{-k-1}$. Now we take logarithmic derivatives:

$$t(\log f(t))' = -(\sigma + k) - (k+1)\frac{\tau/t}{2\pi\nu - \tau/t} = -\sigma - \frac{2\pi k + \frac{\tau}{t\nu}}{2\pi - \frac{\tau}{t\nu}} < -\sigma \leq 0,$$

since, again by $\frac{|\tau|}{t|\nu|} < 2\pi$ and $k \geq 1$, we have $2\pi k + \frac{\tau}{t\nu} > 0$, and, as we said, $2\pi - \frac{\tau}{t\nu} > 0$. □

Lemma 7.4.5 (Estimating an integral)**.** Let $s = \sigma + i\tau$, $\sigma \geq 0$, $\tau \in \mathbb{R}$. Let $\nu \in \mathbb{R} \setminus \{0\}$, $b > a > \frac{|\tau|}{2\pi|\nu|}$. Then

$$\int_a^b t^{-s} e(\nu t) dt = \left.\frac{t^{-\sigma} e(\varphi_\nu(t))}{2\pi i \varphi_\nu'(t)}\right|_a^b + \frac{a^{-\sigma-1}}{2\pi^2} O^*\left(\frac{\sigma}{(\nu-\vartheta)^2} + \frac{|\vartheta|}{|\nu-\vartheta|^3}\right),$$

where $\varphi_\nu(t) = \nu t - \frac{\tau}{2\pi}\log t$ and $\vartheta = \frac{\tau}{2\pi a}$.

Proof. Apply Lemma 7.4.1. Since $\varphi_\nu'(t) = \nu - \tau/(2\pi t)$, we know by Lemma 7.4.4 (with $k = 1$) that $g_1(t) = \frac{t^{-\sigma-1}}{(\varphi_\nu'(t))^2}$ is decreasing on $[a, b]$. We know that $\varphi_\nu'(t) \neq 0$ for $t \geq a$ by $a > \frac{|\tau|}{2\pi|\nu|}$, and so we also know that $g_1(t)$ is continuous for $t \geq a$. Hence, by Lemma 7.4.3,

$$\left|\int_a^b \frac{t^{-\sigma-1}}{2\pi i \varphi_\nu'(t)} e(\varphi_\nu(t)) dt\right| \leq \frac{1}{2\pi} \cdot \frac{|g_1(a)|}{\pi} = \frac{1}{2\pi^2} \frac{a^{-\sigma-1}}{|\nu - \vartheta|^2},$$

since $\varphi_\nu'(a) = \nu - \vartheta$. We remember to include the factor of σ in front of an integral in (7.1).

Since $\varphi_\nu'(t)$ is as above and $\varphi_\nu''(t) = \tau/(2\pi t^2)$, we know by Lemma 7.4.4 (with $k = 2$) that $g_2(t) = \frac{t^{-\sigma}|\varphi_\nu''(t)|}{|\varphi_\nu'(t)|^3} = \frac{|\tau|}{2\pi}\frac{t^{-\sigma-2}}{|\varphi_\nu'(t)|^3}$ is decreasing on $[a, b]$ we also know, as before, that $g_2(t)$ is continuous. Hence, again by Lemma 7.4.3,

$$\left|\int_a^b \frac{t^{-\sigma}\varphi_\nu''(t)}{2\pi i(\varphi_\nu'(t))^2} e(\varphi_\nu(t)) dt\right| \leq \frac{1}{2\pi}\frac{|g_2(a)|}{\pi} = \frac{1}{2\pi^2}\frac{a^{-\sigma-1}|\vartheta|}{|\nu-\vartheta|^3}.$$

□

Lemma 7.4.6 (Estimating an sum)**.** Let $s = \sigma + i\tau$, $\sigma, \tau \in \mathbb{R}$. Let $n \in \mathbb{Z}_{>0}$. Let $a, b \in \mathbb{Z} + \frac{1}{2}$, $b > a > \frac{|\tau|}{2\pi n}$. Write $\varphi_\nu(t) = \nu t - \frac{\tau}{2\pi}\log t$. Then

$$\frac{1}{2}\sum_{\nu=\pm n} \left.\frac{t^{-\sigma} e(\varphi_\nu(t))}{2\pi i \varphi_\nu'(t)}\right|_a^b = \left.\frac{(-1)^n t^{-s} \cdot \frac{\tau}{2\pi t}}{2\pi i\left(n^2 - \left(\frac{\tau}{2\pi t}\right)^2\right)}\right|_a^b.$$

Proof. Since $e(\varphi_\nu(t)) = e(\nu t) t^{-i\tau} = (-1)^\nu t^{-i\tau}$ for any half-integer t and any integer ν,

$$\left.\frac{t^{-\sigma} e(\varphi_\nu(t))}{2\pi i \varphi_\nu'(t)}\right|_a^b = \left.\frac{(-1)^\nu t^{-s}}{2\pi i \varphi_\nu'(t)}\right|_a^b$$

for $\nu = \pm n$. Clearly $(-1)^\nu = (-1)^n$. Since $\varphi_\nu'(t) = \nu - \alpha$ for $\alpha = \frac{\tau}{2\pi t}$,

$$\frac{1}{2}\sum_{\nu=\pm n}\frac{1}{\varphi_\nu'(t)} = \frac{1/2}{n-\alpha} + \frac{1/2}{-n-\alpha} = \frac{-\alpha}{\alpha^2 - n^2} = \frac{\alpha}{n^2 - \alpha^2}.$$

□

It is this easy step that gives us quadratic decay on n. It is just as in the proof of van der Corput's Process B in, say, [13, I.6.3, Thm. 4].

Proposition 7.4.1 (Estimating a Fourier cosine integral)**.** Let $s = \sigma + i\tau$, $\sigma \geq 0$, $\tau \in \mathbb{R}$. Let $a, b \in \mathbb{Z} + \frac{1}{2}$, $b > a > \frac{|\tau|}{2\pi}$. Write $\vartheta = \frac{\tau}{2\pi a}$. Then, for any integer $n \geq 1$,

$$\int_a^b t^{-s} \cos 2\pi nt \, dt = \left(\frac{(-1)^n t^{-s}}{2\pi i} \cdot \frac{\frac{\tau}{2\pi t}}{n^2 - \left(\frac{\tau}{2\pi t}\right)^2} \right) \Bigg|_a^b$$
$$+ \frac{a^{-\sigma-1}}{4\pi^2} O^* \left(\frac{\sigma}{(n-\vartheta)^2} + \frac{\sigma}{(n+\vartheta)^2} + \frac{|\vartheta|}{|n-\vartheta|^3} + \frac{|\vartheta|}{|n+\vartheta|^3} \right).$$

Proof. Write $\cos 2\pi nt = \frac{1}{2}(e(nt) + e(-nt))$. Since $n \geq 1$ and $a > \frac{|\tau|}{2\pi}$, we know that $a > \frac{|\tau|}{2\pi n}$, and so we can apply Lemma 7.4.5 with $\nu = \pm n$. We then apply Lemma 7.4.6 to combine the boundary contributions $|_a^b$ for $\nu = \pm n$. ☐

7.4.2 Approximating zeta(s)

We start with an application of Euler-Maclaurin.

Lemma 7.4.7 (Identity for a partial sum of zeta(s) for integer b)**.** Let $b > 0$, $b \in \mathbb{Z}$. Then, for all $s \in \mathbb{C} \setminus \{1\}$ with $\Re s > 0$,

$$\sum_{n \leq b} \frac{1}{n^s} = \zeta(s) + \frac{b^{1-s}}{1-s} + \frac{b^{-s}}{2} + s \int_b^\infty \left(\{y\} - \frac{1}{2} \right) \frac{dy}{y^{s+1}}. \tag{7.2}$$

Proof. Assume first that $\Re s > 1$. By first-order Euler-Maclaurin,

$$\sum_{n > b} \frac{1}{n^s} = \int_b^\infty \frac{dy}{y^s} + \int_b^\infty \left(\{y\} - \frac{1}{2} \right) d\left(\frac{1}{y^s} \right).$$

Here $\int_b^\infty \frac{dy}{y^s} = -\frac{b^{1-s}}{1-s}$ and $d\left(\frac{1}{y^s}\right) = -\frac{s}{y^{s+1}} dy$. Hence, by $\sum_{n \leq b} \frac{1}{n^s} = \zeta(s) - \sum_{n > b} \frac{1}{n^s}$ for $\Re s > 1$,

$$\sum_{n \leq b} \frac{1}{n^s} = \zeta(s) + \frac{b^{1-s}}{1-s} + \int_b^\infty \left(\{y\} - \frac{1}{2} \right) \frac{s}{y^{s+1}} dy.$$

Since the integral converges absolutely for $\Re s > 0$, both sides extend holomorphically to $\{s \in \mathbb{C} : \Re s > 0, s \neq 1\}$; thus, the equation holds throughout that region. ☐

Lemma 7.4.8 (Identity for a partial sum of zeta(s))**.** Let $b > 0$, $b \in \mathbb{Z} + \frac{1}{2}$. Then, for all $s \in \mathbb{C} \setminus \{1\}$ with $\Re s > 0$,

$$\sum_{n \leq b} \frac{1}{n^s} = \zeta(s) + \frac{b^{1-s}}{1-s} + s \int_b^\infty \left(\{y\} - \frac{1}{2} \right) \frac{dy}{y^{s+1}}. \tag{7.3}$$

Proof. Assume first that $\Re s > 1$. By first-order Euler-Maclaurin and $b \in \mathbb{Z} + \frac{1}{2}$,

$$\sum_{n > b} \frac{1}{n^s} = \int_b^\infty \frac{dy}{y^s} + \int_b^\infty \left(\{y\} - \frac{1}{2} \right) d\left(\frac{1}{y^s} \right).$$

Here $\int_b^\infty \frac{dy}{y^s} = -\frac{b^{1-s}}{1-s}$ and $d\left(\frac{1}{y^s}\right) = -\frac{s}{y^{s+1}} dy$. Hence, by $\sum_{n \leq b} \frac{1}{n^s} = \zeta(s) - \sum_{n > b} \frac{1}{n^s}$ for $\Re s > 1$,

$$\sum_{n \leq b} \frac{1}{n^s} = \zeta(s) + \frac{b^{1-s}}{1-s} + \int_b^\infty \left(\{y\} - \frac{1}{2} \right) \frac{s}{y^{s+1}} dy.$$

Since the integral converges absolutely for $\Re s > 0$, both sides extend holomorphically to $\{s \in \mathbb{C} : \Re s > 0, s \neq 1\}$; thus, the equation holds throughout that region. ☐

Lemma 7.4.9 (Estimate for a partial sum of $\zeta(s)$)**.** Let $b > a > 0$, $b \in \mathbb{Z} + \frac{1}{2}$. Then, for all $s \in \mathbb{C} \setminus \{1\}$ with $\sigma = \Re s > 0$,

$$\sum_{n \le a} \frac{1}{n^s} = -\sum_{a < n \le b} \frac{1}{n^s} + \zeta(s) + \frac{b^{1-s}}{1-s} + O^*\left(\frac{|s|}{2\sigma b^\sigma}\right).$$

Proof. By Lemma 7.4.8, $\sum_{n \le a} = \sum_{n \le b} - \sum_{a < n \le b}$, $\left|\{y\} - \frac{1}{2}\right| \le \frac{1}{2}$ and $\int_b^\infty \frac{dy}{|y^{s+1}|} = \frac{1}{\sigma b^\sigma}$. □

Lemma 7.4.10 (Poisson summation for a partial sum of $\zeta(s)$)**.** Let $a, b \in \mathbb{R} \setminus \mathbb{Z}$, $b > a > 0$. Let $s \in \mathbb{C} \setminus \{1\}$. Define $f : \mathbb{R} \to \mathbb{C}$ by $f(y) = 1_{[a,b]}(y)/y^s$. Then

$$\sum_{a < n \le b} \frac{1}{n^s} = \frac{b^{1-s} - a^{1-s}}{1-s} + \lim_{N \to \infty} \sum_{n=1}^{N} (\hat{f}(n) + \hat{f}(-n)).$$

Proof. Since $a \notin \mathbb{Z}$, $\sum_{a < n \le b} \frac{1}{n^s} = \sum_{n \in \mathbb{Z}} f(n)$. By Poisson summation (as in [10, Thm. D.3])

$$\sum_{n \in \mathbb{Z}} f(n) = \lim_{N \to \infty} \sum_{n=-N}^{N} \hat{f}(n) = \hat{f}(0) + \lim_{N \to \infty} \sum_{n=1}^{N} (\hat{f}(n) + \hat{f}(-n)),$$

where we use the facts that f is in L^1, of bounded variation, and (by $a, b \notin \mathbb{Z}$) continuous at every integer. Now

$$\hat{f}(0) = \int_{\mathbb{R}} f(y) dy = \int_a^b \frac{dy}{y^s} = \frac{b^{1-s} - a^{1-s}}{1-s}.$$

□

We could prove these equations starting from Euler's product for $\sin \pi z$.

Lemma 7.4.11 (Euler/Mittag-Leffler expansion for cosec)**.** Let $z \in \mathbb{C}$, $z \notin \mathbb{Z}$. Then

$$\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{n > 0} (-1)^n \left(\frac{1}{z-n} + \frac{1}{z+n}\right).$$

Proof. Let us start from the Mittag-Leffler expansion $\pi \cot \pi s = \frac{1}{s} + \sum_n \left(\frac{1}{s-n} + \frac{1}{s+n}\right)$.

Applying the trigonometric identity $\cot u - \cot\left(u + \frac{\pi}{2}\right) = \cot u + \tan u = \frac{2}{\sin 2u}$ with $u = \pi z/2$, and letting $s = z/2$, $s = (z+1)/2$, we see that

$$\begin{aligned}
\frac{\pi}{\sin \pi z} &= \frac{\pi}{2} \cot \frac{\pi z}{2} - \frac{\pi}{2} \cot \frac{\pi(z+1)}{2} \\
&= \frac{1/2}{z/2} + \sum_n \left(\frac{1/2}{\frac{z}{2} - n} + \frac{1/2}{\frac{z}{2} + n}\right) - \frac{1/2}{(z+1)/2} - \sum_n \left(\frac{1/2}{\frac{z+1}{2} - n} + \frac{1/2}{\frac{z+1}{2} + n}\right) \\
&= \frac{1}{z} + \sum_n \left(\frac{1}{z-2n} + \frac{1}{z+2n}\right) - \sum_n \left(\frac{1}{z-(2n-1)} + \frac{1}{z+(2n-1)}\right)
\end{aligned}$$

after reindexing the second sum. Regrouping terms again, we obtain our equation. □

Lemma 7.4.12 (Euler/Mittag-Leffler expansion for cosec squared)**.** Let $z \in \mathbb{C}$, $z \notin \mathbb{Z}$. Then

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

Proof. Differentiate the expansion of $\pi \cot \pi z$ term-by-term because it converges uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$. By $(\pi \cot \pi z)' = -\frac{\pi^2}{\sin^2 \pi z}$ and $\left(\frac{1}{z \pm n}\right)' = -\frac{1}{(z\pm n)^2}$, we are done. □

Lemma 7.4.13 (Estimate for an inverse cubic series)**.** For $\vartheta \in \mathbb{R}$ with $0 \le |\vartheta| < 1$,

$$\sum_n \left(\frac{1}{(n-\vartheta)^3} + \frac{1}{(n+\vartheta)^3} \right) \le \frac{1}{(1-|\vartheta|)^3} + 2\zeta(3) - 1.$$

Proof. Since $\frac{1}{(n-\vartheta)^3} + \frac{1}{(n+\vartheta)^3}$ is even, we may replace ϑ by $|\vartheta|$. Then we rearrange the sum:

$$\sum_{n=1}^{\infty} \left(\frac{1}{(n-|\vartheta|)^3} + \frac{1}{(n+|\vartheta|)^3} \right) = \frac{1}{(1-|\vartheta|)^3} + \sum_{n=1}^{\infty} \left(\frac{1}{(n+1-|\vartheta|)^3} + \frac{1}{(n+|\vartheta|)^3} \right).$$

We may write $(n+1-|\vartheta|)^3$, $(n+|\vartheta|)^3$ as $(n+\frac{1}{2}-t)^3$, $(n+\frac{1}{2}+t)^3$ for $t = |\vartheta| - 1/2$. Since $1/u^3$ is convex, $\frac{1}{(n+1/2-t)^3} + \frac{1}{(n+1/2+t)^3}$ reaches its maximum on $[-1/2, 1/2]$ at the endpoints. Hence

$$\sum_{n=1}^{\infty} \left(\frac{1}{(n+1-|\vartheta|)^3} + \frac{1}{(n+|\vartheta|)^3} \right) \le \sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{(n+1)^3} \right) = 2\zeta(3) - 1.$$

□

Lemma 7.4.14 (Estimate for a Fourier sum)**.** Let $s = \sigma + i\tau$, $\sigma \ge 0$, $\tau \in \mathbb{R}$, with $s \ne 1$. Let $b > a > 0$, $a, b \in \mathbb{Z} + \frac{1}{2}$, with $a > \frac{|\tau|}{2\pi}$. Define $f : \mathbb{R} \to \mathbb{C}$ by $f(y) = 1_{[a,b]}(y)/y^s$. Write $\vartheta = \frac{\tau}{2\pi a}$, $\vartheta_- = \frac{\tau}{2\pi b}$. Then

$$\sum_n (\hat{f}(n) + \hat{f}(-n)) = \frac{a^{-s} g(\vartheta)}{2i} - \frac{b^{-s} g(\vartheta_-)}{2i} + O^*\left(\frac{C_{\sigma,\vartheta}}{a^{\sigma+1}} \right)$$

with absolute convergence, where $g(t) = \frac{1}{\sin \pi t} - \frac{1}{\pi t}$ for $t \ne 0$, $g(0) = 0$, and

$$C_{\sigma,\vartheta} = \begin{cases} \frac{\sigma}{2}\left(\frac{1}{\sin^2 \pi\vartheta} - \frac{1}{(\pi\vartheta)^2} \right) + \frac{|\vartheta|}{2\pi^2}\left(\frac{1}{(1-|\vartheta|)^3} + 2\zeta(3) - 1 \right) & \text{for } \vartheta \ne 0, \\ \sigma/6 & \text{for } \vartheta = 0. \end{cases} \tag{7.4}$$

Proof. By Proposition 7.4.1, multiplying by 2 (since $e(-nt) + e(nt) = 2\cos 2\pi nt$),

$$\begin{aligned} \hat{f}(n) + \hat{f}(-n) &= \frac{a^{-s}}{2\pi i} \frac{(-1)^{n+1} 2\vartheta}{n^2 - \vartheta^2} - \frac{b^{-s}}{2\pi i} \frac{(-1)^{n+1} 2\vartheta_-}{n^2 - \vartheta_-^2} \\ &\quad + \frac{a^{-\sigma-1}}{2\pi^2} O^*\left(\frac{\sigma}{(n-\vartheta)^2} + \frac{\sigma}{(n+\vartheta)^2} + \frac{|\vartheta|}{(n-\vartheta)^3} + \frac{|\vartheta|}{(n+\vartheta)^3} \right), \end{aligned} \tag{7.5}$$

where $\vartheta_- = \tau/(2\pi b)$. Note $|\vartheta_-| \le |\vartheta| < 1$. By the Lemma **??**,

$$\sum_n \frac{(-1)^{n+1} 2z}{n^2 - z^2} = \frac{\pi}{\sin \pi z} - \frac{1}{z}$$

for $z \ne 0$, while $\sum_n \frac{(-1)^{n+1} 2z}{n^2 - z^2} = \sum_n 0 = 0$ for $z = 0$. Moreover, by Lemmas 7.4.12 and 7.4.13, for $\vartheta \ne 0$,

$$\sum_n \left(\frac{\sigma}{(n-\vartheta)^2} + \frac{\sigma}{(n+\vartheta)^2} \right) \le \sigma \cdot \left(\frac{\pi^2}{\sin^2 \pi\vartheta} - \frac{1}{\vartheta^2} \right),$$

$$\sum_n \left(\frac{|\vartheta|}{(n-\vartheta)^3} + \frac{|\vartheta|}{(n+\vartheta)^3}\right) \leq |\vartheta| \cdot \left(\frac{1}{(1-|\vartheta|)^3} + 2\zeta(3) - 1\right).$$

If $\vartheta = 0$, then $\sum_n \left(\frac{\sigma}{(n-\vartheta)^2} + \frac{\sigma}{(n+\vartheta)^2}\right) = 2\sigma \sum_{n=1}^{\infty} \frac{1}{n^2} = \sigma \frac{\pi^2}{3}$. □

Proposition 7.4.2 (Approximation of zeta(s) by a partial sum)**.** Let $s = \sigma + i\tau$, $\sigma \geq 0$, $\tau \in \mathbb{R}$, with $s \neq 1$. Let $a \in \mathbb{Z} + \frac{1}{2}$ with $a > \frac{|\tau|}{2\pi}$. Then

$$\zeta(s) = \sum_{n \leq a} \frac{1}{n^s} - \frac{a^{1-s}}{1-s} + c_\vartheta a^{-s} + O^*\left(\frac{C_{\sigma,\vartheta}}{a^{\sigma+1}}\right), \tag{7.6}$$

where $\vartheta = \frac{\tau}{2\pi a}$, $c_\vartheta = \frac{i}{2}\left(\frac{1}{\sin \pi\vartheta} - \frac{1}{\pi\vartheta}\right)$ for $\vartheta \neq 0$, $c_0 = 0$, and $C_{\sigma,\vartheta}$ is as in (7.4).

Proof. Assume first that $\sigma > 0$. Let $b \in \mathbb{Z} + \frac{1}{2}$ with $b > a$, and define $f(y) = \frac{1_{[a,b]}(y)}{y^s}$. By Lemma 7.4.9 and Lemma 7.4.10,

$$\sum_{n \leq a} \frac{1}{n^s} = \zeta(s) + \frac{a^{1-s}}{1-s} - \lim_{N\to\infty} \sum_{n=1}^{N} (\hat{f}(n) + \hat{f}(-n)) + O^*\left(\frac{2|s|}{\sigma b^\sigma}\right).$$

We apply Lemma 7.4.14 to estimate $\lim_{N\to\infty} \sum_{n=1}^{N} (\hat{f}(n) + \hat{f}(-n))$. We obtain

$$\sum_{n \leq a} \frac{1}{n^s} = \zeta(s) + \frac{a^{1-s}}{1-s} - \frac{a^{-s} g(\vartheta)}{2i} + O^*\left(\frac{C_{\sigma,\vartheta}}{a^{\sigma+1}}\right) + \frac{b^{-s} g(\vartheta_-)}{2i} + O^*\left(\frac{2|s|}{\sigma b^\sigma}\right),$$

where $\vartheta_- = \frac{\tau}{2\pi b}$ and $g(t)$ is as in Lemma 7.4.14, and so $-\frac{g(\vartheta)}{2i} = c_\vartheta$. We let $b \to \infty$ through the half-integers, and obtain (7.6), since $b^{-\sigma} \to 0$, $\vartheta_- \to 0$ and $g(\vartheta_-) \to g(0) = 0$ as $b \to \infty$.

Finally, the case $\sigma = 0$ follows since all terms in (7.6) extend continuously to $\sigma = 0$. □

Remark 7.4.1. The term $c_\vartheta a^{-s}$ in (7.6) does not seem to have been worked out before in the literature; the factor of i in c_ϑ was a surprise. For the sake of comparison, let us note that, if $a \geq x$, then $|\vartheta| \leq 1/2\pi$, and so $|c_\vartheta| \leq |c_{\pm 1/2\pi}| = 0.04291\ldots$ and $|C_{\sigma,\vartheta}| \leq |C_{\sigma,\pm 1/2\pi}| \leq 0.176\sigma + 0.246$. While c_ϑ is optimal, $C_{\sigma,\vartheta}$ need not be – but then that is irrelevant for most applications.

Chapter 8

Primary explicit estimates

8.1 Definitions

In this section we define the basic types of primary estimates we will work with in the project.

Key references:
FKS1: Fiori–Kadiri–Swidninsky arXiv:2204.02588
FKS2: Fiori–Kadiri–Swidninsky arXiv:2206.12557

Definition 8.1.1 (Equation (2) of FKS2)**.** $E_\psi(x) = |\psi(x) - x|/x$

Definition 8.1.2 (Definition 1, FKS2)**.** We say that E_ψ satisfies a *classical bound* with parameters A, B, C, R, x_0 if for all $x \geq x_0$ we have

$$E_\psi(x) \leq A \left(\frac{\log x}{R}\right)^B \exp\left(-C \left(\frac{\log x}{R}\right)^{1/2}\right).$$

8.2 A Lemma involving the Möbius Function

In this section we establish a lemma involving sums of the Möbius function.

Definition 8.2.1 (Q)**.** $Q(x)$ is the number of squarefree integers $\leq x$.

Definition 8.2.2 (R)**.** $R(x) = Q(x) - x/\zeta(2)$.

Definition 8.2.3 (M)**.** $M(x)$ is the summatory function of the Möbius function.

Sublemma 8.2.1 (Mobius Lemma 1, initial step)**.** For any $x > 0$,

$$Q(x) = \sum_{k \leq x} M\left(\sqrt{\frac{x}{k}}\right)$$

.

Proof. We compute

$$Q(x) = \sum_{n \leq x} \sum_{d : d^2 | n} \mu(d) = \sum_{k, d : kd^2 \leq x} \mu(d)$$

giving the claim. □

Lemma 8.2.1 (Mobius Lemma 1)**.** For any $x > 0$,

$$R(x) = \sum_{k \le x} M\left(\sqrt{\frac{x}{k}}\right) - \int_0^x M\left(\sqrt{\frac{x}{u}}\right) du. \tag{8.1}$$

Proof. The equality is immediate from Theorem 8.2.1 and exchanging the order of $\sum$ and $\int$, as is justified by $\sum_n |\mu(n)| \int_0^{x/n^2} du \le \sum_n x/n^2 < \infty$)

$$\int_0^x M\left(\sqrt{\frac{x}{u}}\right) du = \int_0^x \sum_{n \le \sqrt{\frac{x}{u}}} \mu(n) du = \sum_n \mu(n) \int_0^{\frac{x}{n^2}} du = x \sum_n \frac{\mu(n)}{n^2} = \frac{x}{\zeta(2)}.$$

□

Since our sums start from 1, the sum $\sum_{k \le K}$ is empty for $K = 0$.

Sublemma 8.2.2 (Mobius Lemma 2 - first step)**.** For any $K \le x$,

$$\sum_{k \le x} M\left(\sqrt{\frac{x}{k}}\right) = \sum_{k \le K} M\left(\sqrt{\frac{x}{k}}\right) + \sum_{K < k \le x+1} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} M\left(\sqrt{\frac{x}{k}}\right) du.$$

Proof. This is just splitting the sum at K. □

Sublemma 8.2.3 (Mobius Lemma 2 - second step)**.** For any $K \le x$, for $f(u) = M(\sqrt{x/u})$,

$$\sum_{K < k \le x+1} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f(u) du = \int_{K+\frac{1}{2}}^{\lfloor x \rfloor + \frac{3}{2}} f(u) du = \int_{K+\frac{1}{2}}^{x} f(u) du,$$

Proof. This is just splitting the integral at K, since $f(u) = M(\sqrt{x/u}) = 0$ for $x > u$. □

Lemma 8.2.2 (Mobius Lemma 2)**.** For any $x > 0$ and any integer $K \ge 0$,

$$\begin{aligned} R(x) = & \sum_{k \le K} M\left(\sqrt{\frac{x}{k}}\right) - \int_0^{K+\frac{1}{2}} M\left(\sqrt{\frac{x}{u}}\right) du \\ & - \sum_{K < k \le x+1} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \left(M\left(\sqrt{\frac{x}{u}}\right) - M\left(\sqrt{\frac{x}{k}}\right)\right) du \end{aligned} \tag{8.2}$$

Proof. We split into two cases. If $K > x$, the second line of (8.2) is empty, and the first one equals (8.1), by $M(t) = 0$ for $t < 1$, so (8.2) holds.

Now suppose that $K \le x$. Then we combine Sublemma 8.2.2 and Sublemma 8.2.3 with Lemma 8.2.1 to give the claim. □

8.3 The estimates of Fiori, Kadiri, and Swidinsky

In this section we establish the primary results of Fiori, Kadiri, and Swidinsky [6].

Theorem 8.3.1 (FKS Theorem 2.7)**.** Let H_0 denote a verification height for RH. Let $10^9/H_0 \leq k \leq 1$, $t > 0$, $H \in [1002, H_0)$, $\alpha > 0$, $\delta \geq 1$, $\eta_0 = 0.23622$, $1 + \eta_0 \leq \mu \leq 1 + \eta$, and $\eta \in (\eta_0, 1/2)$ be fixed. Let $\sigma > 1/2 + d/\log H_0$. Then for any $T \geq H_0$, one has

$$N(\sigma, T) \leq (T-H)\log T/(2\pi d) * \log(1+CC_1(\log(kT))^{2\sigma}(\log T)^{4(1-\sigma)}T^{8/3(1-\sigma)}/(T-H)) + CC_2 * \log^2 T/2\pi d$$

and

$$N(\sigma, T) \leq \frac{CC_1}{2\pi d}(\log kT)^{2\sigma}(\log T)^{5-4*\sigma}T^{8/3(1-\sigma)} + CC_2 * \log^2 T/2\pi d$$

.

Proof. ☐

Definition 8.3.1 (FKS Corollary 2.9)**.** For each $\sigma_1, \sigma_2, \tilde{c}_1, \tilde{c}_2$ given in Table 8, we have $N(\sigma, T) \leq \tilde{c}_1 T^{p(\sigma)} \log^{q(\sigma)} + \tilde{c}_2 \log^2 T$ for $\sigma_1 \leq \sigma \leq \sigma_2$ with $p(\sigma) = 8/3(1-\sigma)$ and $q(\sigma) = 5 - 2\sigma$.

Theorem 8.3.2 (FKS Lemma 2.1)**.** If $|N(T) - (T/2\pi \log(T/2\pi e) + 7/8)| \leq R(T)$ then $\sum_{U \leq \gamma < V} 1/\gamma \leq B_1(U, V)$.

Proof. ☐

Theorem 8.3.3 (FKS Corollary 2.3)**.** For each pair T_0, S_0 in Table 1 we have, for all $V > T_0$, $\sum_{0 < \gamma < V} 1/\gamma < S_0 + B_1(T_0, V)$.

Proof. ☐

Theorem 8.3.4 (FKS Lemma 2.5)**.** Let $T_0 \geq 2$ and $\gamma > 0$. Assume that there exist c_1, c_2, p, q, T_0 for which one has a zero density bound. Assume $\sigma \geq 5/8$ and $T_0 \leq U < V$. Then $s_0(\sigma, U, V) \leq B_0(\sigma, U, V)$.

Proof. ☐

Theorem 8.3.5 (FKS Remark 2-6-a)**.** $\Gamma(3, x) = (x^2 + 2(x+1))e^{-x}$.

Proof. ☐

Theorem 8.3.6 (FKS Remark 2-6-b)**.** For $s > 1$, one has $\Gamma(s, x) \sim x^{s-1}e^{-x}$.

Proof. ☐

Theorem 8.3.7 (FKS Theorem 3.1)**.** Let $x > e^{50}$ and $50 < T < x$. Then $E_\psi(x) \leq \sum_{|\gamma| < T} |x^{\rho - 1}/\rho| + 2\log^2 x/T$.

Proof. ☐

Theorem 8.3.8 (FKS Theorem 3.2)**.** For any $\alpha \in (0, 1/2]$ and $\omega \in [0, 1]$ there exist M, x_M such that for $\max(51, \log x) < T < (x^\alpha - 2)/5$ and some $T^* \in [T, 2.45T]$,

$$|\psi(x) - (x - \sum_{|\gamma| \leq T^*} x^\rho/\rho)| \leq Mx/T * log^{1-\omega} x$$

for all $x \geq x_M$.

Proof. ☐

Theorem 8.3.9 (FKS Proposition 3.4)**.** Let $x > e^{50}$ and $3\log x < T < \sqrt{x}/3$. Then $E_\psi(x) \leq \sum_{|\gamma|<T} |x^{\rho-1}/\rho| + 2\log^2 x/T$.

Proof. ☐

Theorem 8.3.10 (FKS Proposition 3.6)**.** Let $\sigma_1 \in (1/2, 1)$ and let (T_0, S_0) be taken from Table 1. Then $\Sigma_0^{\sigma_1} \leq 2x^{-1/2}(S_0 + B_1(T_0, T)) + (x_1^{\sigma_1 - 1} - x^{-1/2})B_1(H_0, T)$.

Proof. ☐

Theorem 8.3.11 (FKS equation (3.13))**.** $\Sigma_a^b = 2 * \sum_{H_a \leq \gamma < T; a \leq \beta < b} \frac{x^{\beta-1}}{\gamma}$.

Proof. ☐

Theorem 8.3.12 (FKS Remark 3.7)**.** If $\sigma < 1 - 1/R\log H_0$ then $H_\sigma = H_0$.

Proof. ☐

Theorem 8.3.13 (FKS Proposition 3.8)**.** Let $N \geq 2$ be an integer. If $5/8 \leq \sigma_1 < \sigma_2 \leq 1$, $T \geq H_0$, then $\Sigma_{\sigma_1}^{\sigma_2} \leq 2x^{-(1-\sigma_1)+(\sigma_2-\sigma_1/N)}B_0(\sigma_1, H_{\sigma_1}, T) + 2x^{(1-\sigma_1)}(1 - x^{-(\sigma_2-\sigma_1)/N})\sum_{n=1}^{N-1} B_0(\sigma^{(n)}, H^{(n)}, T)x^{(\sigma_2-\sigma_1)(n-}$

Proof. ☐

Theorem 8.3.14 (FKS Corollary 3.10)**.** If $\sigma_1 \geq 0.9$ then $\Sigma_{\sigma_1}^{\sigma_2} \leq 0.00125994x^{\sigma_2 - 1}$.

Proof. ☐

Theorem 8.3.15 (FKS Proposition 3.11)**.** Let $5/8 < \sigma_2 \leq 1$, $t_0 = t_0(\sigma_2, x) = \max(H_{\sigma_2}, \exp(\sqrt{\log x}/R))$ and $T > 0$. Let $K \geq 2$ and consider a strictly increasing sequence $(t_k)_{k=0}^K$ such that $t_k = T$. Then $\Sigma_{\sigma_2}^1 \leq 2N(\sigma_2, T)x^{-1/R\log t_0}/t_0$ and $\Sigma_{\sigma_2}^1 \leq 2((\sum_{k=1}^{K-1} N(\sigma_2, t_k)(x^{-1/R\log t_{k-1}}/t_{k-1} - x^{-1/(R\log t_k)}/t_k)) + x^{-1/R\log t_{K-1}}/t_{K-1}N(\sigma_2, T))$.

Proof. ☐

Theorem 8.3.16 (FKS Corollary 3.12)**.** Let $5/8 < \sigma_2 \leq 1$, $t_0 = t_0(\sigma_2, x) = \max\left(H_{\sigma_2}, \exp\left(\sqrt{\frac{\log x}{R}}\right)\right)$, $T > t_0$. Let $K \geq 2$, $\lambda = (T/t_0)^{1/K}$, and consider $(t_k)_{k=0}^K$ the sequence given by $t_k = t_0\lambda^k$. Then

$$\Sigma_{\sigma_2}^1 = 2 \sum_{\substack{0<\gamma<T \\ \sigma_2 \leq \beta < 1}} \frac{x^{\beta-1}}{\gamma} \leq \varepsilon_4(x, \sigma_2, K, T),$$

where

$$\varepsilon_4(x, \sigma_2, K, T) = 2\sum_{k=1}^{K-1} \frac{x^{-\frac{1}{R\log t_k}}}{t_k}\left(\tilde{N}(\sigma_2, t_{k+1}) - \tilde{N}(\sigma_2, t_k)\right) + 2\tilde{N}(\sigma_2, t_1)\frac{x^{-\frac{1}{R(\log t_0)}}}{t_0},$$

and $\tilde{N}(\sigma, T)$ satisfy (ZDB) $N(\sigma, T) \leq \tilde{N}(\sigma, T)$.

Proof. ☐

Theorem 8.3.17 (FKS Proposition 3-14)**.** Fix $K \geq 2$ and $c > 1$, and set t_0, T, and σ_2 as functions of x defined by

$$t_0 = t_0(x) = \exp\left(\sqrt{\frac{\log x}{R}}\right), \quad T = t_0^c, \quad \text{and} \quad \sigma_2 = 1 - \frac{2}{R \log t_0}. \tag{8.3}$$

Then, with $\varepsilon_4(x, \sigma_2, K, T)$ as defined in (3.22), we have that as $x \to \infty$,

$$\varepsilon_4(x, \sigma_2, K, T) = (1 + o(1)) C \frac{(\log t_0)^{3 + \frac{4}{R \log t_0}}}{t_0^2}, \quad \text{with } C = 2c_1 e^{\frac{16w_1}{3R}} w_1^3, \text{ and } w_1 = 1 + \frac{c-1}{K}, \tag{8.4}$$

where c_1 is an admissible value for (ZDB) on some interval $[\sigma_1, 1]$. Moreover, both $\varepsilon_4(x, \sigma_2, K, T)$ and $\frac{\varepsilon_4(x,\sigma_2,K,T)t_0^2}{(\log t_0)^3}$ are decreasing in x for $x > \exp(Re^2)$.

Proof. □

Theorem 8.3.18 (FKS Theorem 1.1)**.** For any x_0 with $\log x_0 > 1000$, and all $0.9 < \sigma_2 < 1$, $2 \leq c \leq 30$, and $N, K \geq 1$ the formula $\varepsilon(x_0) := \varepsilon(x_0, \sigma_2, c, N, K)$ as defined in (4.1) gives an effectively computable bound

$$E_\psi(x) \leq \varepsilon(x_0) \quad \text{for all } x \geq x_0.$$

Proof. □

Theorem 8.3.19 (FKS Theorem 1.1b)**.** Moreover, a collection of values, $\varepsilon(x_0)$ computed with well chosen parameters are provided in Table 5.

Proof. □

Theorem 8.3.20 (FKS Lemma 5.2)**.** For all $0 < \log x \leq 2100$ we have that

$$E_\psi(x) \leq 2(\log x)^{3/2} \exp\left(-0.8476836\sqrt{\log x}\right).$$

Proof. □

Theorem 8.3.21 (FKS Lemma 5.3)**.** For all $2100 < \log x \leq 200000$ we have that

$$E_\psi(x) \leq 9.22022(\log x)^{3/2} \exp\left(-0.8476836\sqrt{\log x}\right).$$

Proof. □

Theorem 8.3.22 (FKS Theorem 1.2b)**.** If $\log x_0 \geq 1000$ then we have an admissible bound for E_ψ with the indicated choice of $A(x_0)$, $B = 3/2$, $C = 2$, and $R = 5.5666305$.

Proof. □

Theorem 8.3.23 (FKS1 Corollary 1.3)**.** For all x > 2 we have $E_\psi(x) \leq 121.096(\log x/R)^{3/2} \exp(-2\sqrt{\log x/R})$ with $R = 5.5666305$.

Proof. □

Theorem 8.3.24 (FKS1 Corollary 1.4)**.** For all x > 2 we have $E_\psi(x) \leq 9.22022(\log x)^{3/2} \exp(-0.8476836\sqrt{\log x})$.

Proof. TODO. □

8.4 Summary of results

In this section we list some papers that we plan to incorporate into this section in the future, and list some results that have not yet been moved into dedicated paper sections.

References to add:

None yet.

Chapter 9

Secondary explicit estimates

9.1 Definitions

In this section we define the basic types of secondary estimates we will work with in the project. Key references:
FKS1: Fiori–Kadiri–Swidninsky arXiv:2204.02588
FKS2: Fiori–Kadiri–Swidninsky arXiv:2206.12557

Definition 9.1.1 (pi)**.** $\pi(x)$ is the number of primes less than or equal to x.

Definition 9.1.2 (li and Li)**.** $\mathrm{li}(x) = \int_0^x \frac{dt}{\log t}$ and $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$.

Definition 9.1.3 (theta)**.** $\theta(x) = \sum_{p \leq x} \log p$ where the sum is over primes p.

Definition 9.1.4 (Equation (1) of FKS2)**.** $E_\pi(x) = |\pi(x) - \mathrm{Li}(x)|/\mathrm{Li}(x)$

Definition 9.1.5 (Equation (2) of FKS2)**.** $E_\theta(x) = |\theta(x) - x|/x$

Definition 9.1.6 (Definition 1, FKS2)**.** We say that E_θ satisfies a *classical bound* with parameters A, B, C, R, x_0 if for all $x \geq x_0$ we have

$$E_\theta(x) \leq A \left(\frac{\log x}{R}\right)^B \exp\left(-C \left(\frac{\log x}{R}\right)^{1/2}\right).$$

Similarly for E_π.

Definition 9.1.7 (Definition 1, FKS2)**.** We say that E_π satisfies a *classical bound* with parameters A, B, C, R, x_0 if for all $x \geq x_0$ we have

$$E_\pi(x) \leq A \left(\frac{\log x}{R}\right)^B \exp\left(-C \left(\frac{\log x}{R}\right)^{1/2}\right).$$

9.2 The prime number bounds of Rosser and Schoenfeld

In this section we formalize the prime number bounds of Rosser and Schoenfeld [11].

Theorem 9.2.1 (A medium version of the prime number theorem)**.** $\vartheta(x) = x + O(x/\log^2 x)$.

Proof. This in principle follows by establishing an analogue of Theorem 5.0.1, using mediumPNT in place of weakPNT. ☐

Definition 9.2.1 (Meissel-Mertens constant B)**.** $B := \lim_{x\to\infty}\left(\sum_{p\le x}\frac{1}{p} - \log\log x\right)$.

Definition 9.2.2 (Mertens constant E)**.** $E := \lim_{x\to\infty}\left(\sum_{p\le x}\frac{\log p}{p} - \log x\right)$.

Sublemma 9.2.1 (The Chebyshev function is Stieltjes)**.** The function $\vartheta(x) = \sum_{p\le x}\log p$ defines a Stieltjes function (monotone and right continuous).

Proof. Trivial ☐

Sublemma 9.2.2 (RS-prime display before (4.13))**.** $\sum_{p\le x} f(p) = \int_2^x \frac{f(y)}{\log y}\, d\vartheta(y)$.

Proof. This follows from the definition of the Stieltjes integral. ☐

Sublemma 9.2.3 (RS equation (4.13))**.** $\sum_{p\le x} f(p) = \frac{f(x)\vartheta(x)}{\log x} - \int_2^x \vartheta(x)\frac{d}{dy}(\frac{f(y)}{\log y})\, dy$.

Proof. Follows from Sublemma 9.2.2 and integration by parts. ☐

Sublemma 9.2.4 (RS equation (4.14))**.**

$$\sum_{p\le x} f(p) = \int_2^x \frac{f(y)\ dy}{\log y} + \frac{2f(2)}{\log 2}$$

$$+\frac{f(x)(\vartheta(x)-x)}{\log x} - \int_2^x (\vartheta(y)-y)\frac{d}{dy}\frac{d}{dy}(\frac{f(y)}{\log y})\ dy.$$

Proof. Follows from Sublemma 9.2.3 and integration by parts. ☐

Sublemma 9.2.5 (RS equation (4.16))**.**

$$L_f := \frac{2f(2)}{\log 2} - \int_2^\infty (\vartheta(y)-y)\frac{d}{dy}(\frac{f(y)}{\log y})\ dy.$$

Sublemma 9.2.6 (RS equation (4.15))**.**

$$\sum_{p\le x} f(p) = \int_2^x \frac{f(y)\ dy}{\log y} + L_f$$

$$+\frac{f(x)(\vartheta(x)-x)}{\log x} + \int_x^\infty (\vartheta(y)-y)\frac{d}{dy}\frac{d}{dy}(\frac{f(y)}{\log y})\ dy.$$

Proof. Follows from Sublemma 9.2.4 and Definition 9.2.5. ☐

Sublemma 9.2.7 (RS equation (4.17))**.**

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(y)\ dy}{y\log^2 y}.$$

Proof. Follows from Sublemma 9.2.3 applied to $f(t) = 1$. ☐

Sublemma 9.2.8 (RS equation (4.18))**.**

$$\sum_{p \leq x} \frac{1}{p} = \frac{\vartheta(x)}{x \log x} + \int_2^x \frac{\vartheta(y)(1 + \log y)\ dy}{y^2 \log^2 y}.$$

Proof. Follows from Sublemma 9.2.3 applied to $f(t) = 1/t$. ☐

Theorem 9.2.2 (RS equation (4.19) and Mertens' second theorem)**.**

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + B + \frac{\vartheta(x) - x}{x \log x}$$
$$- \int_2^x \frac{(\vartheta(y) - y)(1 + \log y)\ dy}{y^2 \log^2 y}.$$

Proof. Follows from Sublemma 9.2.3 applied to $f(t) = 1/t$. One can also use this identity to demonstrate convergence of the limit defining B. ☐

Theorem 9.2.3 (RS equation (4.19) and Mertens' first theorem)**.**

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + E + \frac{\vartheta(x) - x}{x}$$
$$- \int_2^x \frac{(\vartheta(y) - y)\ dy}{y^2}.$$

Proof. Follows from Sublemma 9.2.3 applied to $f(t) = \log t/t$. Convergence will need Theorem 9.2.1. ☐

9.3 Tools from BKLNW

In this file we record the results from [1]. -

9.4 The implications of FKS2

In this file we record the implications in the paper [7] that allow one to convert primary bounds on E_ψ into secondary bounds on E_π, E_θ.

Remark 9.4.1 (Remark in FKS2 Section 1.1)**.** $\mathrm{li}(x) - \mathrm{Li}(x) = \mathrm{li}(2)$.

Proof. This follows directly from the definitions of li and Li. ☐

Definition 9.4.1 (g function, FKS2 (16))**.** For any $a, b, c, x \in \mathbb{R}$ we define $g(a, b, c, x) := x^{-a}(\log x)^b \exp(c(\log x)^{1/2})$.

Sublemma 9.4.1 (FKS2 equation (17))**.** For any $2 \leq x_0 < x$ one has

$$(\pi(x) - \mathrm{Li}(x)) - (\pi(x_0) - \mathrm{Li}(x_0)) = \frac{\theta(x) - x}{\log x} - \frac{\theta(x_0) - x_0}{\log x_0} + \int_{x_0}^x \frac{\theta(t) - t}{t \log^2 t} dt.$$

Proof. This follows from Sublemma 9.2.7. ☐

Sublemma 9.4.2 (FKS2 Sublemma 10-1)**.** We have

$$\frac{d}{dx} g(a,b,c,x) = \left(-a\log(x) + b + \frac{c}{2}\sqrt{\log(x)}\right) x^{-a-1}(\log(x))^{b-1}\exp(c\sqrt{\log(x)}).$$

Proof. This follows from straightforward differentiation. ☐

Sublemma 9.4.3 (FKS2 Sublemma 10-2)**.** $\frac{d}{dx} g(a,b,c,x)$ is negative when $-au^2 + \frac{c}{2}u + b < 0$, where $u = \sqrt{\log(x)}$.

Proof. Clear from previous sublemma. ☐

Lemma 9.4.1 (FKS2 Lemma 10a)**.** If $a > 0$, $c > 0$ and $b < -c^2/16a$, then $g(a,b,c,x)$ decreases with x.

Proof. We apply Lemma 9.4.3. There are no roots when $b < -\frac{c^2}{16a}$, and the derivative is always negative in this case. ☐

Lemma 9.4.2 (FKS2 Lemma 10b)**.** For any $a > 0$, $c > 0$ and $b \geq -c^2/16a$, $g(a,b,c,x)$ decreases with x for $x > \exp((\frac{c}{4a} + \frac{1}{2a}\sqrt{\frac{c^2}{4} + 4ab})^2)$.

Proof. We apply Lemma 9.4.3. If $a > 0$, there are two real roots only if $\frac{c^2}{4} + 4ab \geq 0$ or equivalently $b \geq -\frac{c^2}{16a}$, and the derivative is negative for $u > \frac{\frac{c}{2} + \sqrt{\frac{c^2}{4} + 4ab}}{2a}$. ☐

Lemma 9.4.3 (FKS2 Lemma 10c)**.** If $c > 0$, $g(0,b,c,x)$ decreases with x for $\sqrt{\log x} > -2b/c$.

Proof. We apply Lemma 9.4.3. If $a = 0$, it is negative when $u < \frac{-2b}{c}$. ☐

Corollary 9.4.1 (FKS2 Corollary 11)**.** If $B \geq 1 + C^2/16R$ then $g(1, 1-B, C/\sqrt{R}, x)$ is decreasing in x.

Proof. This follows from Lemma 9.4.1 applied with $a = 1$, $b = 1 - B$ and $c = C/\sqrt{R}$. ☐

Definition 9.4.2 (Dawson function, FKS2 (19))**.** The Dawson function $D_+ : \mathbb{R} \to \mathbb{R}$ is defined by the formula $D_+(x) := e^{-x^2}\int_0^x e^{t^2}\ dt$.

Remark 9.4.2 (FKS2 remark after Corollary 11)**.** The Dawson function has a single maximum at $x \approx 0.942$, after which the function is decreasing.

Proof. The Dawson function satisfies the differential equation $F'(x) + 2xF(x) = 1$ from which it follows that the second derivative satisfies $F''(x) = -2F(x) - 2x(-2xF(x) + 1)$, so that at every critical point (where we have $F(x) = \frac{1}{2x}$) we have $F''(x) = -\frac{1}{x}$. It follows that every positive critical value gives a local maximum, hence there is a unique such critical value and the function decreases after it. Numerically one may verify this is near 0.9241 see https://oeis.org/ A133841. ☐

Lemma 9.4.4 (FKS2 Lemma 12)**.** Suppose that E_θ satisfies an admissible classical bound with parameters A, B, C, R, x_0. Then, for all $x \geq x_0$,

$$\int_{x_0}^{x} |\frac{E_\theta(t)}{\log^2 t} dt| \leq \frac{2A}{R^B} x m(x_0, x)\exp(-C\sqrt{\frac{\log x}{R}}) D_+(\sqrt{\log x} - \frac{C}{2\sqrt{R}})$$

where

$$m(x_0, x) = \max((\log x_0)^{(2B-3)/2}, (\log x)^{(2B-3)/2}).$$

Proof. Since $\varepsilon_{\theta,\mathrm{asymp}}(t)$ provides an admissible bound on $\theta(t)$ for all $t \geq x_0$, we have

$$\int_{x_0}^{x} \left|\frac{\theta(t)-t}{t(\log(t))^2}\right| dt \leq \int_{x_0}^{x} \frac{\varepsilon_{\theta,\mathrm{asymp}}(t)}{(\log(t))^2} = \frac{A_\theta}{R^B}\int_{x_0}^{x} (\log(t))^{B-2} \exp\left(-C\sqrt{\frac{\log(t)}{R}}\right) dt.$$

We perform the substitution $u = \sqrt{\log(t)}$ and note that $u^{2B-3} \leq m(x_0, x)$ as defined in (21). Thus the above is bounded above by

$$\frac{2A_\theta m(x_0,x)}{R^B}\int_{\sqrt{\log(x_0)}}^{\sqrt{\log(x)}} \exp\left(u^2 - \frac{Cu}{\sqrt{R}}\right) du.$$

Then, by completing the square $u^2 - \frac{Cu}{\sqrt{R}} = \left(u - \frac{C}{2\sqrt{R}}\right)^2 - \frac{C^2}{4R}$ and doing the substitution $v = u - \frac{C}{2\sqrt{R}}$, the above becomes

$$\frac{2A_\theta m(x_0,x)}{R^B} \exp\left(-\frac{C^2}{4R}\right)\int_{\sqrt{\log(x_0)}-\frac{C}{2\sqrt{R}}}^{\sqrt{\log(x)}-\frac{C}{2\sqrt{R}}} \exp(v^2)\, dv.$$

Now we have

$$\begin{aligned}\int_{\sqrt{\log(x_0)}-\frac{C}{2\sqrt{R}}}^{\sqrt{\log(x)}-\frac{C}{2\sqrt{R}}} \exp(v^2)\, dv &\leq \int_{0}^{\sqrt{\log(x)}-\frac{C}{2\sqrt{R}}} \exp(v^2)\, dv \\ &= x\exp\left(\frac{C^2}{4R}\right)\exp\left(-C\sqrt{\frac{\log(x)}{R}}\right) D_+\left(\sqrt{\log(x)} - \frac{C}{2\sqrt{R}}\right).\end{aligned}$$

Combining the above completes the proof. □

Theorem 9.4.1 (FKS2 Proposition 13)**.** Suppose that A_ψ, B, C, R, x_0 give an admissible bound for E_ψ. If $B > C^2/8R$, then A_θ, B, C, R, x_0 give an admissible bound for E_θ, where

$$A_\theta = A_\psi(1 + \nu_{asymp}(x_0))$$

with

$$\nu_{asymp}(x_0) = \frac{1}{A_\psi}(\frac{R}{\log x_0})^B \exp(C\sqrt{\frac{\log x_0}{R}})(a_1(\log x_0)x_0^{-1/2} + a_2(\log x_0)x_0^{-2/3}).$$

Proof. □

Theorem 9.4.2 (FKS2 Corollary 14)**.** We have an admissible bound for E_θ with $A = 121.0961$, $B = 3/2$, $C = 2$, $R = 5.5666305$, $x_0 = 2$.

Proof. □

Definition 9.4.3 (mu asymptotic function, FKS2 (9))**.** For $x_0, x_1 > 0$, we define

$$mu_{asymp}(x_0,x_1) := \frac{x_0\log(x_1)}{\epsilon_{\theta,asymp}(x_1)x_1\log(x_0)}\left|\frac{\pi(x_0)-\mathrm{Li}(x_0)}{x_0/\log x_0} - \frac{\theta(x_0)-x_0}{x_0}\right| + \frac{2D_+(\sqrt{\log(x_1)} - \frac{C}{2\sqrt{R}}}{\sqrt{\log x_1}}$$

.

Definition 9.4.4 (FKS2 Definition 5)**.** Let $x_0 > 2$. We say a (step) function $\varepsilon_{\diamond,num}(x_0)$ gives an admissible numerical bound for $E_\diamond(x)$ if $E_\diamond(x) \leq \varepsilon_{\diamond,num}(x_0)$ for all $x \geq x_0$.

Theorem 9.4.3 (FKS2 Remark 7)**.** If

$$\frac{d}{dx}\frac{\log x}{x}\left(Li(x) - \frac{x}{\log x} - Li(x_1) + \frac{x_1}{\log x_1}\right)|_{x_2} \geq 0$$

then $\mu_{num,1}(x_0, x_1, x_2) < \mu_{num,2}(x_0, x_1)$.

Proof. □

Theorem 9.4.4 (FKS2 Remark 15)**.** If $\log x_0 \geq 1000$ then we have an admissible bound for E_θ with the indicated choice of $A(x_0)$, $B = 3/2$, $C = 2$, and $R = 5.5666305$.

Proof. □

Theorem 9.4.5 (FKS2 Theorem 3)**.** If $B \geq \max(3/2, 1 + C^2/16R)$, $x_0 > 0$, and one has an admissible asymptotic bound with parameters A, B, C, x_0 for E_θ, and

$$x_1 \geq \max(x_0, \exp((1 + \frac{C}{2\sqrt{R}}))^2),$$

then

$$E_\pi(x) \leq \epsilon_{\theta,asymp}(x_1)(1 + \mu_{asymp}(x_0, x_1))$$

for all $x \geq x_1$. In other words, we have an admissible bound with parameters $(1 + \mu_{asymp}(x_0, x_1))A, B, C, x_1$ for E_π.

Proof. □

Theorem 9.4.6 (FKS2 Proposition 17)**.** Let $x > x_0 > 2$. IF $E_\psi(x) \leq \varepsilon_{\psi,num}(x_0)$, then

$$-\varepsilon_{\theta,num}(x_0) \leq \frac{\theta(x) - x}{x} \leq \varepsilon_{\psi,num}(x_0) \leq \varepsilon_{\theta,num}(x_0)$$

where

$$\varepsilon_{\theta,num}(x_0) = \varepsilon_{\psi,num}(x_0) + 1.00000002(x_0^{-1/2} + x_0^{-2/3} + x_0^{-4/5}) + 0.94(x_0^{-3/4} + x_0^{-5/6} + x_0^{-9/10})$$

Proof. □

Theorem 9.4.7 (FKS2 Lemma 19)**.** Let $x_1 > x_0 \geq 2$, $N \in \mathbb{N}$, and let $(b_i)_{i=1}^N$ be a finite partition of $[x_0, x_1]$. Then

$$|\int_{x_0}^{x_1} \frac{\theta(t) - t}{t \log^2 t}\, dt| \leq \sum_{i=1}^{N-1} \varepsilon_{\theta,num}(e^{b_i})(Li(e^{b_{i+1}}) - Li(e^{b_i}) + \frac{e^{b_i}}{b_i} - \frac{e^{b_{i+1}}}{b_{i+1}}).$$

Proof. □

Theorem 9.4.8 (FKS2 Lemma 20)**.** Assume $x \geq 6.58$. Then $Li(x) - \frac{x}{\log x}$ is strictly increasing and $Li(x) - \frac{x}{\log x} > \frac{x - 6.58}{\log^2 x} > 0$.

Proof. □

Theorem 9.4.9 (FKS2 Theorem 6)**.** Let $x_0 > 0$ be chosen such that $\pi(x_0)$ and $\theta(x_0)$ are computable, and let $x_1 \geq \max(x_0, 14)$. Let $\{b_i\}_{i=1}^N$ be a finite partition of $[\log x_0, \log x_1]$, with $b_1 = \log x_0$ and $b_N = \log x_1$, and suppose that $\varepsilon_{\theta,\mathrm{num}}$ gives computable admissible numerical bounds for $x = \exp(b_i)$, for each $i = 1, \dots, N$. For $x_1 \leq x_2 \leq x_1 \log x_1$, we define

$$\mu_{num}(x_0, x_1, x_2) = \frac{x_0 \log x_1}{\varepsilon_{\theta,num}(x_0) x_1 \log x_0} \left| \frac{\pi(x_0) - \mathrm{Li}(x_0)}{x_0/\log x_0} - \frac{\theta(x_0) - x_0}{x_0} \right|$$

$$+ \frac{\log x_1}{\varepsilon_{theta,num}(x_1) x_1} \sum_{i=1}^{N-1} \varepsilon_{\theta,num}(\exp(b_i)) \left(Li(e^{b_{i+1}}) - Li(e^{b_i}) + \frac{e^{b_i}}{b_i} - \frac{e^{b_{i+1}}}{b_{i+1}} \right)$$

$$+ \frac{\log x_2}{x_2} \left(Li(x_2) - \frac{x_2}{\log x_2} - Li(x_1) + \frac{x_1}{\log x_1} \right)$$

and for $x_2 > x_1 \log x_1$, including the case $x_2 = \infty$, we define

$$\mu_{num}(x_0, x_1, x_2) = \frac{x_0 \log x_1}{\varepsilon_{\theta,num}(x_1) x_1 \log x_0} \left| \frac{\pi(x_0) - \mathrm{Li}(x_0)}{x_0/\log x_0} - \frac{\theta(x_0) - x_0}{x_0} \right|$$

$$+ \frac{\log x_1}{\varepsilon_{\theta,num}(x_1) x_1} \sum_{i=1}^{N-1} \varepsilon_{\theta,num}(\exp(b_i)) \left(Li(e^{b_{i+1}}) - Li(e^{b_i}) + \frac{e^{b_i}}{b_i} - \frac{e^{b_{i+1}}}{b_{i+1}} \right)$$

$$+ \frac{1}{\log x_1 + \log\log x_1 - 1}.$$

Then, for all $x_1 \leq x \leq x_2$ we have

$$E_\pi(x) \leq \varepsilon_{\pi,num}(x_1, x_2) := \varepsilon_{\theta,num}(x_1)(1 + \mu_{num}(x_0, x_1, x_2)).$$

Proof. □

Theorem 9.4.10 (FKS2 Corollary 8)**.** Let $\{b'_i\}_{i=1}^M$ be a set of finite subdivisions of $[\log(x_1), \infty)$, with $b'_1 = \log(x_1)$ and $b'_M = \infty$. Define

$$\varepsilon_{\pi,num}(x_1) := \max_{1 \leq i \leq M-1} \varepsilon_{\pi,num}(\exp(b'_i), \exp(b'_{i+1})).$$

Then $E_\pi(x) \leq \varepsilon_{\pi,num}(x_1)$ for all $x \geq x_1$.

Proof. □

Theorem 9.4.11 (FKS2 Corollary 21)**.** Let $B \geq \max(\frac{3}{2}, 1 + \frac{C^2}{16R})$. Let $x_0, x_1 > 0$ with $x_1 \geq \max(x_0, \exp((1 + \frac{C}{2\sqrt{R}})^2))$. If E_ψ satisfies an admissible classical bound with parameters A_ψ, B, C, R, x_0, then E_π satisfies an admissible classical bound with A_π, B, C, R, x_1, where

$$A_\pi = (1 + \nu_{asymp}(x_0))(1 + \mu_{asymp}(x_0, x_1))A_\psi$$

for all $x \geq x_0$, where

$$|E_\theta(x)| \leq \varepsilon_{\theta,asymp}(x) := A(1 + \mu_{asymp}(x_0, x)) \exp(-C\sqrt{\frac{\log x}{R}})$$

where

$$\nu_{asymp}(x_0) = \frac{1}{A_\psi}(\frac{R}{\log x_0})^B \exp(C\sqrt{\frac{\log x_0}{R}})(a_1(\log x_0)x_0^{-1/2} + a_2(\log x_0)x_0^{-2/3})$$

and

$$\mu_{asymp}(x_0, x_1) = \frac{x_0 \log x_1}{\varepsilon_{\theta,asymp}(x_1) x_1 \log x_0} |E_\pi(x_0) - E_\theta(x_0)| + \frac{2D_+(\sqrt{\log x} - \frac{C}{2\sqrt{R}})}{\sqrt{\log x_1}}.$$

Proof. ☐

Theorem 9.4.12 (FKS2 Corollary 22)**.** One has

$$|\pi(x) - \mathrm{Li}(x)| \leq 9.2211 x \sqrt{\log x} \exp(-0.8476 \sqrt{\log x})$$

for all $x \geq 2$.

Proof. ☐

Theorem 9.4.13 (FKS2 Corollary 23)**.** A_π, B, C, x_0 as in Table 6 give an admissible asymptotic bound for E_π with $R = 5.5666305$.

Proof. ☐

Theorem 9.4.14 (FKS2 Corollary 24)**.** We have the bounds $E_\pi(x) \leq B(x)$, where $B(x)$ is given by Table 7.

Proof. ☐

Theorem 9.4.15 (FKS2 Corollary 26)**.** One has

$$|\pi(x) - \mathrm{Li}(x)| \leq 0.4298 \frac{x}{\log x}$$

for all $x \geq 2$.

Proof. ☐

9.5 Summary of results

Here we list some papers that we plan to incorporate into this section in the future, and list some results that have not yet been moved into dedicated paper sections.

References to add:

Dusart: `https://piyanit.nl/wp-content/uploads/2020/10/art_10.1007_s11139-016-9839-4.pdf`

PT: D. J. Platt and T. S. Trudgian, The error term in the prime number theorem, Math. Comp. 90 (2021), no. 328, 871–881.

JY: D. R. Johnston, A. Yang, Some explicit estimates for the error term in the prime number theorem, arXiv:2204.01980.

Theorem 9.5.1 (PT Corollary 2)**.** One has

$$|\pi(x) - \mathrm{Li}(x)| \leq 235 x (\log x)^{0.52} \exp(-0.8 \sqrt{\log x})$$

for all $x \geq \exp(2000)$.

Proof. ☐

Theorem 9.5.2 (JY Corollary 1.3)**.** One has

$$|\pi(x) - \mathrm{Li}(x)| \leq 9.59x(\log x)^{0.515} \exp(-0.8274\sqrt{\log x})$$

for all $x \geq 2$.

Proof. □

Theorem 9.5.3 (JY Theorem 1.4)**.** One has

$$|\pi(x) - \mathrm{Li}(x)| \leq 0.028x(\log x)^{0.801} \exp(-0.1853 \log^{3/5} x/(\log \log x)^{1/5}))$$

for all $x \geq 2$.

Proof. □

TODO: input other results from JY

Theorem 9.5.4 (Dusart Proposition 5.4)**.** There exists a constant X_0 (one may take $X_0 = 89693$) with the following property: for every real $x \geq X_0$, there exists a prime p with

$$x < p \leq x\Big(1 + \frac{1}{\log^3 x}\Big).$$

Proof. □

TODO: input other results from Dusart

Chapter 10

Tertiary explicit estimates

10.1 The least common multiple sequence is not highly abundant for large n

10.1.1 Problem statement and notation

Definition 10.1.1. $\sigma(n)$ is the sum of the divisors of n.

Definition 10.1.2. A positive integer N is called *highly abundant* (HA) if

$$\sigma(N) > \sigma(m)$$

for all positive integers $m < N$, where $\sigma(n)$ denotes the sum of the positive divisors of n.

Informally, a highly abundant number has an unusually large sum of divisors.

Definition 10.1.3. For each integer $n \geq 1$, define

$$L_n := \mathrm{lcm}(1, 2, \ldots, n).$$

We call $(L_n)_{n\geq 1}$ the *least common multiple sequence*.

Clearly L_n has a lot of divisors, and numerical experiments for small n suggests that L_n is often highly abundant. This leads to the following question:

> **Original MathOverflow question.** Is it true that every value in the sequence $L_n = \mathrm{lcm}(1, 2, \ldots, n)$ is highly abundant? Equivalently,
>
> $$\{L_n : n \geq 1\} \subseteq HA?$$

Somewhat surprisingly, the answer is *no*: not every L_n is highly abundant.

It has previously been verified in Lean that L_n is highly aboundant for $n \leq 70$, $81 \leq n \leq 96$, $125 \leq n \leq 148$, $169 \leq n \leq 172$, and not highly abondant for all other $n \leq 10^{10}$. The arguments here establish the non-highly-abundance of L_n for all $n \geq 89683^2$ sufficiently large n, thus completing the determination in Lean of all n for which L_n is highly abundant. This argument was taken from this MathOverflow answer.

10.1.2 A general criterion using three medium primes and three large primes

The key step in the proof is to show that, if one can find six primes $p_1, p_2, p_3, q_1, q_2, q_3$ obeying a certain inequality, then one can find a competitor $M < L_n$ to L_n with $\sigma(M) > \sigma(L_n)$, which will demonstrate that L_n is not highly abundant. More precisely:

Definition 10.1.4. In the next few subsections we assume that $n \geq 1$ and that $p_1, p_2, p_3, q_1, q_2, q_3$ are primes satisfiying

$$\sqrt{n} < p_1 < p_2 < p_3 < q_1 < q_2 < q_3 < n$$

and the key criterion

$$\prod_{i=1}^{3}\Big(1 + \frac{1}{q_i}\Big) \leq \left(\prod_{i=1}^{3}\Big(1 + \frac{1}{p_i(p_i+1)}\Big)\right)\Big(1 + \frac{3}{8n}\Big)\left(1 - \frac{4p_1p_2p_3}{q_1q_2q_3}\right). \tag{10.1}$$

NOTE: In the Lean formalization of this argument, we index the primes from 0 to 2 rather than from 1 to 3.

Lemma 10.1.1. We have $4p_1p_2p_3 < q_1q_2q_3$.

Proof. Obvious from the non-negativity of the left-hand side of (10.1). □

10.1.3 Factorisation of L_n and construction of a competitor

Lemma 10.1.2 (Factorisation of L_n)**.** There exists a positive integer L' such that

$$L_n = q_1q_2q_3\, L'$$

and each prime q_i divides L_n exactly once and does not divide L'.

Proof. Since $q_i < n$, the prime q_i divides L_n exactly once (as $q_i^2 > n$). Hence we may write $L_n = q_1q_2q_3L'$ where L' is the quotient obtained by removing these prime factors. By construction, $q_i \nmid L'$ for each i. □

Lemma 10.1.3 (Normalised divisor sum for L_n)**.** Let L' be as in Lemma 10.1.2. Then

$$\frac{\sigma(L_n)}{L_n} = \frac{\sigma(L')}{L'}\prod_{i=1}^{3}\Big(1 + \frac{1}{q_i}\Big). \tag{10.2}$$

Proof. Use the multiplicativity of $\sigma(\cdot)$ and the fact that each q_i occurs to the first power in L_n. Then

$$\sigma(L_n) = \sigma(L')\prod_{i=1}^{3}\sigma(q_i) = \sigma(L')\prod_{i=1}^{3}(1 + q_i).$$

Dividing by $L_n = L'\prod_{i=1}^{3} q_i$ gives

$$\frac{\sigma(L_n)}{L_n} = \frac{\sigma(L')}{L'}\prod_{i=1}^{3}\frac{1+q_i}{q_i} = \frac{\sigma(L')}{L'}\prod_{i=1}^{3}\Big(1 + \frac{1}{q_i}\Big).$$

□

Lemma 10.1.4. There exist integers $m \geq 0$ and r satisfying $0 < r < 4p_1p_2p_3$ and

$$q_1q_2q_3 = 4p_1p_2p_3m + r$$

Proof. This is division with remainder. □

Definition 10.1.5. With m, r as above, define the competitor

$$M := 4p_1p_2p_3mL'.$$

Lemma 10.1.5 (Basic properties of M)**.** With notation as above, we have:

1. $M < L_n$.
2. $$1 < \frac{L_n}{M} = \Big(1 - \frac{r}{q_1q_2q_3}\Big)^{-1} < \Big(1 - \frac{4p_1p_2p_3}{q_1q_2q_3}\Big)^{-1}.$$

Proof. The first item is by construction of the division algorithm. For the second, note that

$$L_n = q_1q_2q_3L' = (4p_1p_2p_3m + r)L' > 4p_1p_2p_3mL' = M,$$

since $r > 0$. For the third,

$$\frac{L_n}{M} = \frac{4p_1p_2p_3m + r}{4p_1p_2p_3m} = 1 + \frac{r}{4p_1p_2p_3m} = \Big(1 - \frac{r}{4p_1p_2p_3m + r}\Big)^{-1} = \Big(1 - \frac{r}{q_1q_2q_3}\Big)^{-1}.$$

Since $0 < r < 4p_1p_2p_3$ and $4p_1p_2p_3 < q_1q_2q_3$, we have

$$0 < \frac{r}{q_1q_2q_3} < \frac{4p_1p_2p_3}{q_1q_2q_3},$$

so

$$\Big(1 - \frac{r}{q_1q_2q_3}\Big)^{-1} < \Big(1 - \frac{4p_1p_2p_3}{q_1q_2q_3}\Big)^{-1}.$$

□

10.1.4 A sufficient condition

We give a sufficient condition for $\sigma(M) \geq \sigma(L_n)$.

Lemma 10.1.6 (A sufficient inequality)**.** Suppose

$$\frac{\sigma(M)}{M}\Big(1 - \frac{4p_1p_2p_3}{q_1q_2q_3}\Big) \;\geq\; \frac{\sigma(L_n)}{L_n}.$$

Then $\sigma(M) \geq \sigma(L_n)$, and so L_n is not highly abundant.

Proof. By Lemma 10.1.5,

$$\frac{L_n}{M} < \Big(1 - \frac{4p_1p_2p_3}{q_1q_2q_3}\Big)^{-1}.$$

Hence

$$\frac{\sigma(M)}{M} \geq \frac{\sigma(L_n)}{L_n}\Big(1 - \frac{4p_1p_2p_3}{q_1q_2q_3}\Big)^{-1} > \frac{\sigma(L_n)}{L_n} \cdot \frac{M}{L_n}.$$

Multiplying both sides by M gives

$$\sigma(M) > \sigma(L_n) \cdot \frac{M}{L_n}$$

and hence

$$\sigma(M) \geq \sigma(L_n),$$

since $M/L_n < 1$ and both sides are integers. □

Combining Lemma 10.1.6 with Lemma 10.1.3, we see that it suffices to bound $\sigma(M)/M$ from below in terms of $\sigma(L')/L'$:

Lemma 10.1.7 (Reduction to a lower bound for $\sigma(M)/M$)**.** If

$$\frac{\sigma(M)}{M} \geq \frac{\sigma(L')}{L'} \left(\prod_{i=1}^{3} \Big(1 + \frac{1}{p_i(p_i+1)}\Big) \right) \Big(1 + \frac{3}{8n}\Big), \tag{10.3}$$

then L_n is not highly abundant.

Proof. Insert (10.3) and (10.2) into the desired inequality and compare with the assumed bound (10.1); this is a straightforward rearrangement. □

10.1.5 Conclusion of the criterion

Lemma 10.1.8 (Lower bound for $\sigma(M)/M$)**.** With notation as above,

$$\frac{\sigma(M)}{M} \geq \frac{\sigma(L')}{L'} \left(\prod_{i=1}^{3} \Big(1 + \frac{1}{p_i(p_i+1)}\Big) \right) \Big(1 + \frac{3}{8n}\Big).$$

Proof. By multiplicativity, we have

$$\frac{\sigma(M)}{M} = \frac{\sigma(L')}{L'} \prod_p \frac{1 + p^{-1} + \cdots + p^{-\nu_p(M)}}{1 + p^{-1} + \cdots + p^{-\nu_p(L')}}.$$

The contribution of $p = p_i$ is

$$\frac{(1 + p_i^{-1} + p_i^{-2})}{1 + p_i^{-1}} = 1 + \frac{1}{p_i(p_i+1)}.$$

The contribution of $p = 2$ is

$$\frac{1 + 2^{-1} + \cdots + 2^{-k-2}}{1 + 2^{-1} + \cdots + 2^{-k}},$$

where k is the largest integer such that $2^k \leq n$. A direct calculation yields

$$\frac{(1 + 2^{-1} + \cdots + 2^{-k-2})}{1 + 2^{-1} + \cdots + 2^{-k}} = \frac{2^{k+3} - 1}{2^{k+3} - 4} = 1 + \frac{3}{2^{k+3} - 4},$$

Finally, since $2^k \leq n < 2^{k+1}$, we have $2^{k+3} < 8n$, so

$$\frac{3}{2^{k+3} - 4} \geq \frac{3}{8n},$$

So the contribution from the prime 2 is at least $1 + 3/(8n)$.

Finally, the contribution of all other primes is at least 1. □

We have thus completed the key step of demonstrating a sufficient criterion to establish that L_n is not highly abundant:

Theorem 10.1.1. Let $n \geq 1$. Suppose that primes $p_1, p_2, p_3, q_1, q_2, q_3$ satisfy

$$\sqrt{n} < p_1 < p_2 < p_3 < q_1 < q_2 < q_3 < n$$

and the key criterion (10.1). Then L_n is not highly abundant.

Proof. By Lemma 10.1.8, the condition (10.3) holds. By Lemma 10.1.7 this implies

$$\frac{\sigma(M)}{M}\Big(1 - \frac{4p_1p_2p_3}{q_1q_2q_3}\Big) \geq \frac{\sigma(L_n)}{L_n}.$$

Applying Lemma 10.1.6, we obtain $\sigma(M) \geq \sigma(L_n)$ with $M < L_n$, so L_n cannot be highly abundant. □

Remark 10.1.1. Analogous arguments allow other pairs (c, α) in place of $(4, 3/8)$, such as $(2, 1/4)$, $(6, 17/36)$, $(30, 0.632\ldots)$; or even $(1, 0)$ provided more primes are used on the p-side than the q-side to restore an asymptotic advantage.

10.1.6 Choice of six primes p_i, q_i for large n

To finish the proof we need to locate six primes $p_1, p_2, p_3, q_1, q_2, q_3$ obeying the required inequality. Here we will rely on the prime number theorem of Dusart [5].

Lemma 10.1.9 (Choice of medium primes p_i)**.** Let $n \geq X_0^2$. Set $x := \sqrt{n}$. Then there exist primes p_1, p_2, p_3 with

$$p_i \leq x\Big(1 + \frac{1}{\log^3 x}\Big)^i$$

and $p_1 < p_2 < p_3$. Moreover, $\sqrt{n} < p_1$

Proof. Apply Theorem 9.5.4 successively with $x, x(1 + 1/\log^3 x), x(1 + 1/\log^3 x)^2$, keeping track of the resulting primes and bounds. For n large and $x = \sqrt{n}$, we have $\sqrt{n} < p_1$ as soon as the first interval lies strictly above $\sqrt{n}$; this can be enforced by taking n large enough. □

Lemma 10.1.10 (Choice of large primes q_i)**.** Let $n \geq X_0^2$. Then there exist primes $q_1 < q_2 < q_3$ with

$$q_{4-i} \geq n\Big(1 + \frac{1}{\log^3 \sqrt{n}}\Big)^{-i}$$

for $i = 1, 2, 3$, and $q_1 < q_2 < q_3 < n$.

Proof. Apply Theorem 9.5.4 with suitable values of x slightly below n, e.g. $x = n(1 + 1/\log^3 \sqrt{n})^{-i}$, again keeping track of the intervals. For n large enough, these intervals lie in $(\sqrt{n}, n)$ and contain primes q_i with the desired ordering. □

10.1.7 Bounding the factors in (10.1)

Lemma 10.1.11 (Bounds for the q_i-product)**.** With p_i, q_i as in Lemmas 10.1.9 and 10.1.10, we have

$$\prod_{i=1}^{3}\Big(1+\frac{1}{q_i}\Big) \le \prod_{i=1}^{3}\Big(1+\frac{(1+\frac{1}{\log^3\sqrt{n}})^i}{n}\Big). \tag{10.4}$$

Proof. By Lemma 10.1.10, each q_i is at least

$$n\Big(1+\frac{1}{\log^3\sqrt{n}}\Big)^{-j}$$

for suitable indices j, so $1/q_i$ is bounded above by

$$\frac{(1+\frac{1}{\log^3\sqrt{n}})^i}{n}$$

after reindexing. Multiplying the three inequalities gives (10.4). □

Lemma 10.1.12 (Bounds for the p_i-product)**.** With p_i as in Lemma 10.1.9, we have for large n

$$\prod_{i=1}^{3}\Big(1+\frac{1}{p_i(p_i+1)}\Big) \ge \prod_{i=1}^{3}\Big(1+\frac{1}{(1+\frac{1}{\log^3\sqrt{n}})^{2i}(n+\sqrt{n})}\Big). \tag{10.5}$$

Proof. By Lemma 10.1.9, $p_i \le \sqrt{n}(1+1/\log^3\sqrt{n})^i$. Hence

$$p_i^2 \le n(1+\tfrac{1}{\log^3\sqrt{n}})^{2i} \quad \text{and} \quad p_i+1 \le p_i(1+1/\sqrt{n}) \le (1+1/\sqrt{n})p_i.$$

From these one gets

$$p_i(p_i+1) \le (1+\tfrac{1}{\log^3\sqrt{n}})^{2i}(n+\sqrt{n}),$$

and hence

$$\frac{1}{p_i(p_i+1)} \ge \frac{1}{(1+\frac{1}{\log^3\sqrt{n}})^{2i}(n+\sqrt{n})}.$$

Taking $1+\cdot$ and multiplying over $i=1,2,3$ gives (10.5). □

Lemma 10.1.13 (Lower bound for the product ratio p_i/q_i)**.** With p_i, q_i as in Lemmas 10.1.9 and 10.1.10, we have

$$1-\frac{4p_1p_2p_3}{q_1q_2q_3} \ge 1-\frac{4(1+\frac{1}{\log^3\sqrt{n}})^{12}}{n^{3/2}}. \tag{10.6}$$

Proof. We have $p_i \le \sqrt{n}(1+1/\log^3\sqrt{n})^i$, so

$$p_1p_2p_3 \le n^{3/2}(1+\tfrac{1}{\log^3\sqrt{n}})^6.$$

Similarly, $q_i \ge n(1+1/\log^3\sqrt{n})^{-i}$, so

$$q_1q_2q_3 \ge n^3(1+\tfrac{1}{\log^3\sqrt{n}})^{-6}.$$

Combining,

$$\frac{p_1p_2p_3}{q_1q_2q_3} \le \frac{n^{3/2}\big(1+\frac{1}{\log^3\sqrt{n}}\big)^6}{n^3\big(1+\frac{1}{\log^3\sqrt{n}}\big)^{-6}} = \frac{\big(1+\frac{1}{\log^3\sqrt{n}}\big)^{12}}{n^{3/2}}.$$

This implies (10.6). □

10.1.8 Reduction to a small epsilon-inequality

Lemma 10.1.14 (Uniform bounds for large n)**.** For all $n \ge X_0^2 = 89693^2$ we have

$$\frac{1}{\log^3\sqrt{n}} \le 0.000675, \qquad \frac{1}{n^{3/2}} \le \frac{1}{89693}\cdot\frac{1}{n}.$$

and

$$\frac{1}{n+\sqrt{n}} \ge \frac{1}{1+1/89693}\cdot\frac{1}{n}.$$

Proof. This is a straightforward calculus and monotonicity check: the left-hand sides are decreasing in n for $n \ge X_0^2$, and equality (or the claimed upper bound) holds at $n = X_0^2$. One can verify numerically or symbolically. □

Lemma 10.1.15 (Polynomial approximation of the inequality)**.** For $0 \le \varepsilon \le 1/89693^2$, we have

$$\prod_{i=1}^{3}(1+1.000675^i\varepsilon) \le \Big(1+3.01\varepsilon+3.01\varepsilon^2+1.01\varepsilon^3\Big),$$

and

$$\prod_{i=1}^{3}\Big(1+\frac{\varepsilon}{1.000675^{2i}}\frac{1}{1+1/89693}\Big)\Big(1+\frac{3}{8}\varepsilon\Big)\Big(1-\frac{4\times 1.000675^{12}}{89693}\varepsilon\Big) \ge 1+3.36683\varepsilon-0.01\varepsilon^2.$$

Proof. Expand each finite product as a polynomial in ε, estimate the coefficients using the bounds in Lemma 10.1.14, and bound the tails by simple inequalities such as

$$(1+C\varepsilon)^k \le 1+kC\varepsilon+...$$

for small ε. All coefficients can be bounded numerically in a rigorous way; this step is a finite computation that can be checked mechanically. □

Lemma 10.1.16 (Final polynomial comparison)**.** For $0 \le \varepsilon \le 1/89693^2$, we have

$$1+3.01\varepsilon+3.01\varepsilon^2+1.01\varepsilon^3 \le 1+3.36683\varepsilon-0.01\varepsilon^2.$$

Proof. This is equivalent to

$$3.01\varepsilon+3.01\varepsilon^2+1.01\varepsilon^3 \le 3.36683\varepsilon-0.01\varepsilon^2,$$

or

$$0 \le (3.36683-3.01)\varepsilon-(3.01+0.01)\varepsilon^2-1.01\varepsilon^3.$$

Factor out ε and use that $0 < \varepsilon \le 1/89693^2$ to check that the resulting quadratic in ε is nonnegative on this interval. Again, this is a finite computation that can be verified mechanically. □

Proposition 10.1.1 (Verification of (10.1) for large n)**.** For every integer $n \geq X_0^2 = 89693^2 \approx 8.04 \times 10^9$, the primes p_i, q_i constructed in Lemmas 10.1.9 and 10.1.10 satisfy the inequality (10.1).

Proof. Combine Lemma 10.1.11, Lemma 10.1.12, and Lemma 10.1.13. Then apply Lemma 10.1.14 and set $\varepsilon = 1/n$. The products are bounded by the expressions in Lemma 10.1.15, and the inequality in Lemma 10.1.16 then ensures that (10.1) holds. ☐

10.1.9 Conclusion for large n

Theorem 10.1.2 (Non-highly abundant for large n)**.** For every integer $n \geq 89693^2$, the integer L_n is not highly abundant.

Proof. By Proposition 10.1.1, there exist primes $p_1, p_2, p_3, q_1, q_2, q_3$ satisfying the hypotheses of Theorem 10.1.1. Hence L_n is not highly abundant. ☐

10.2 Erdos problem 392

The proof here is adapted from `https://www.erdosproblems.com/forum/thread/392#post-2696` which in turn is inspired by the arguments in `https://arxiv.org/abs/2503.20170`.

Definition 10.2.1. We work with (approximate) factorizations $a_1 \dots a_t$ of a factorial $n!$.

Definition 10.2.2. The waste of a factorizations $a_1 \dots a_t$ is defined as $\sum_i \log(n/a_i)$.

Definition 10.2.3. The balance of a factorization $a_1 \dots a_t$ at a prime p is defined as the number of times p divides $a_1 \dots a_t$, minus the number of times p divides $n!$.

Lemma 10.2.1. If a factorization has zero total imbalance, then it exactly factors $n!$.

Proof. ☐

Lemma 10.2.2. The waste of a factorization is equal to $t \log n - \log n!$, where t is the number of elements.

Proof. ☐

Definition 10.2.4. The score of a factorization (relative to a cutoff parameter L) is equal to its waste, plus $\log p$ for every surplus prime p, $\log(n/p)$ for every deficit prime above L, $\log L$ for every deficit prime below L and an additional $\log n$ if one is not in total balance.

Lemma 10.2.3. If one is in total balance, then the score is equal to the waste.

Proof. ☐

Sublemma 10.2.1. If there is a prime p in surplus, one can remove it without increasing the score.

Proof. Locate a factor a_i that contains the surplus prime p, then replace a_i with a_i/p. ☐

Sublemma 10.2.2. If there is a prime p in deficit larger than L, one can remove it without increasing the score.

Proof. Add an additional factor of p to the factorization. ☐

Sublemma 10.2.3. If there is a prime p in deficit less than L, one can remove it without increasing the score.

Proof. Without loss of generality we may assume that one is not in the previous two situations, i.e., wlog there are no surplus primes and all primes in deficit are at most L. If all deficit primes multiply to n or less, add that product to the factorization (this increases the waste by at most $\log n$, but we save a $\log n$ from now being in balance). Otherwise, greedily multiply all primes together while staying below n until one cannot do so any further; add this product to the factorization, increasing the waste by at most $\log L$. ☐

Lemma 10.2.4. One can bring any factorization into balance without increasing the score.

Proof. Apply strong induction on the total imbalance of the factorization and use the previous three sublemmas. ☐

Proposition 10.2.1. Starting from any factorization f, one can find a factorization f' in balance whose cardinality is at most $\log n!$ plus the score of f, divided by $\log n$.

Proof. Combine Lemma 10.2.4, Lemma 10.2.3, and Lemma 10.2.2. ☐

Definition 10.2.5. Now let M, L be additional parameters with $n > L^2$; we also need the minor variant $\lfloor n/L \rfloor > \sqrt{n}$.

Definition 10.2.6. We perform an initial factorization by taking the natural numbers between $n - n/M$ (inclusive) and n (exclusive) repeated M times, deleting those elements that are not n/L-smooth (i.e., have a prime factor greater than or equal to n/L).

Sublemma 10.2.4. The number of elements in this initial factorization is at most n.

Proof. ☐

Lemma 10.2.5. The total waste in this initial factorization is at most $n \log \frac{1}{1-1/M}$.

Proof. ☐

Sublemma 10.2.5. A large prime $p \geq n/L$ cannot be in surplus.

Proof. No such prime can be present in the factorization. ☐

Sublemma 10.2.6. A large prime $p \geq n/L$ can be in deficit by at most n/p.

Proof. This is the number of times p can divide $n!$. ☐

Sublemma 10.2.7. A medium prime $\sqrt{n} < p \leq n/L$ can be in surplus by at most M.

Proof. Routine computation using Legendre's formula. ☐

Sublemma 10.2.8. A medium prime $\sqrt{n} < p \leq n/L$ can be in deficit by at most M.

Proof. The number of times p divides $a_1 \dots a_t$ is at least $M\lfloor n/Mp \rfloor \geq n/p - M$ (note that the removal of the non-smooth numbers does not remove any multiples of p). Meanwhile, the number of times p divides $n!$ is at most n/p. ☐

Sublemma 10.2.9. A small prime $p \leq \sqrt{n}$ can be in surplus by at most $M \log n$.

Proof. Routine computation using Legendre's formula, noting that at most $\log n / \log 2$ powers of p divide any given number up to n. ☐

Sublemma 10.2.10. A small prime $L < p \leq \sqrt{n}$ can be in deficit by at most $M \log n$.

Proof. Routine computation using Legendre's formula, noting that at most $\log n / \log 2$ powers of p divide any given number up to n. ☐

Sublemma 10.2.11. A tiny prime $p \leq L$ can be in deficit by at most $M \log n + ML\pi(n)$.

Proof. In addition to the Legendre calculations, one potentially removes factors of the form plq with $l \leq L$ and $q \leq n$ a prime up to M times each, with at most L copies of p removed at each factor. ☐

Proposition 10.2.2. The initial score is bounded by

$$n \log(1-1/M)^{-1} + \sum_{p \leq n/L} M \log n + \sum_{p \leq \sqrt{n}} M \log^2 n / \log 2 + \sum_{n/L < p \leq n} \frac{n}{p} \log \frac{n}{p} + \sum_{p \leq L} (M \log n + ML\pi(n)) \log L.$$

Proof. Combine Lemma 10.2.5, Sublemma 10.2.5, Sublemma 10.2.6, Sublemma 10.2.7, Sublemma 10.2.8, Sublemma 10.2.9, Sublemma 10.2.10, and Sublemma 10.2.11, and combine $\log p$ and $\log(n/p)$ to $\log n$. ☐

Sublemma 10.2.12. If M is sufficiently large depending on ε, then $n \log(1-1/M)^{-1} \leq \varepsilon n$.

Proof. Use the fact that $\log(1 - 1/M)^{-1}$ goes to zero as $M \to \infty$. ☐

Sublemma 10.2.13. If L is sufficiently large depending on M, ε, and n sufficiently large depending on L, then $\sum_{p \leq n/L} M \log n \leq \varepsilon n$.

Proof. Use the prime number theorem (or the Chebyshev bound). ☐

Sublemma 10.2.14. If n sufficiently large depending on M, ε, then $\sum_{p \leq \sqrt{n}} M \log^2 n / \log 2 \leq \varepsilon n$.

Proof. Crude estimation. ☐

Lemma 10.2.6.

$$\pi(n) = o(n) \quad \text{as } n \to \infty.$$

Proof. Given $\varepsilon > 0$, choose $a \neq 0$ with $\varphi(a)/a < \varepsilon/2$ (using $\prod_{p \leq n}(1 - 1/p) \to 0$). For $n \geq a + 2$,

$$\pi(n) \leq \frac{\varphi(a)}{a} \cdot n + \varphi(a) + \pi(a + 1) + 1.$$

Since $\varphi(a)/a < \varepsilon/2$, for n large enough the constant terms are absorbed, giving $\pi(n) < \varepsilon n$. ☐

Sublemma 10.2.15. If n sufficiently large depending on L, ε, then $\sum_{n/L < p \leq n} \frac{n}{p} \log \frac{n}{p} \leq \varepsilon n$.

Proof. Bound $\frac{n}{p}$ by L and use the prime number theorem (or the Chebyshev bound). ☐

Sublemma 10.2.16. For all $n \geq 2$, one has

$$\pi(n) \leq \sqrt{n} + \frac{2n \log 4}{\log n}.$$

Proof. By Chebyshev's bound, $\prod_{p \leq n} p \leq 4^n$, so $\sum_{p \leq n} \log p \leq n \log 4$. The number of primes $p \leq \sqrt{n}$ is trivially at most $\sqrt{n}$. For primes $p > \sqrt{n}$, we have $\log p > \frac{1}{2} \log n$, hence

$$(\pi(n) - \pi(\sqrt{n})) \cdot \tfrac{1}{2} \log n < \sum_{\sqrt{n} < p \leq n} \log p \leq n \log 4,$$

giving $\pi(n) - \pi(\sqrt{n}) < \frac{2n \log 4}{\log n}$. Adding $\pi(\sqrt{n}) \leq \sqrt{n}$ yields the result. □

Sublemma 10.2.17. If n sufficiently large depending on M, L, ε, then $\sum_{p \leq L} (M \log n + ML\pi(n)) \log L \leq \varepsilon n$.

Proof. Use the prime number theorem (or the Chebyshev bound). □

Proposition 10.2.3. The score of the initial factorization can be taken to be $o(n)$.

Proof. Pick M large depending on ε, then L sufficiently large depending on M, ε, then n sufficiently large depending on M, L, ε, so that the bounds in Sublemma 10.2.12, Sublemma 10.2.13, Sublemma 10.2.14, Sublemma 10.2.15, and Sublemma 10.2.17 each contribute at most $(\varepsilon/5)n$. Then use Proposition 10.2.2. □

Theorem 10.2.1. One can find a balanced factorization of $n!$ with cardinality at least $n - n/\log n - o(n/\log n)$.-

Proof. Combine Proposition 10.2.3 with Proposition 10.2.1 and the Stirling approximation. □

Theorem 10.2.2. One can find a factor $n!$ into at least $n/2 - n/2 \log n - o(n/\log n)$ numbers of size at most n^2.-

Proof. Group the factorization arising in Theorem 10.2.1 into pairs, using Lemma 10.2.1. □

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