An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 9: Finite Sums and Boundedness of Convergent Sequences

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1 Finite Sums: Each Term Bounded by the Total

When working with finite sums of nonnegative quantities, a fundamental property emerges: any individual term cannot exceed the total sum. For sequences of real numbers, this translates to the fact that the absolute value of any term is bounded by the sum of all absolute values up to that point.

This result is particularly useful when we need to control individual terms using global information about sums. The proof employs mathematical induction, which is natural since we're dealing with sums over finite ranges.

1.1 New Tools

1.1.1 Summation Notation: range

The notation $\sum_{k \in \text{range } N}$ means summing as k goes from 0 to $N-1$ (which has N terms total). This is a standard way to express finite sums in formal mathematics.

1.1.2 Sum Range Successor: sum_range_succ

For a function $f : \mathbb{N} \to \mathbb{R}$ and natural number N, we have:

$$\sum_{n \in \text{range } (N+1)} f(n) = \sum_{n \in \text{range } N} f(n) + f(N)$$

This allows us to peel off the last term of a sum, which is essential for inductive proofs.

1.1.3 Nonnegativity of Sums: sum_nonneg

If a function is nonnegative everywhere, then its finite sum is also nonnegative. More precisely, if $f(k) \geq 0$ for all k in the range, then $\sum_k f(k) \geq 0$.

1.1.4 Proof by Contradiction: contradiction

If your hypotheses lead to a logical contradiction (for example, if one hypothesis states $n < 0$ where $n : \mathbb{N}$ is a natural number), then the `contradiction` tactic closes the goal automatically.

1.2 The Mathematical Statement

Theorem (TermLeSum): If $a : \mathbb{N} \to \mathbb{R}$ is a sequence and N is a natural number, then for all $n < N$, we have:

$$|a(n)| \leq \sum_{k=0}^{N-1} |a(k)|$$

1.3 Strategic Approach

The key insight is to use induction on N from the very beginning, rather than introducing n first. This is counterintuitive since the statement begins with $\forall n < N$, but the inductive structure makes the proof much cleaner.

For the base case $N = 0$, there are no natural numbers $n < 0$, so the statement is vacuously true (leading to a contradiction when we try to assume such an n exists).

For the inductive step, assuming the result holds for N, we prove it for $N + 1$ by splitting into two cases: either $n < N$ (where we use the inductive hypothesis) or $n = N$ (where we use the nonnegativity of sums).

1.4 Lean Solution

```
Statement TermLeSum (a : ℕ → ℝ) (N : ℕ) :
    ∀ n < N, |a n| ≤ ∑ k ∈ range N, |a k| := by
induction' N with N hN
intro n hn
contradiction
intro n hn
have f1 : ∑ k ∈ range (N + 1), |a k| = (∑ k ∈ range N, |
   a k|) + |a N| := by apply sum_range_succ
rewrite [f1]
by_cases hn' : n < N
specialize hN n hn'
have f1' : 0 ≤ |a N| := by bound
linarith [f1', hN]
have f2 : n = N := by bound
rewrite [f2]
have f3 : ∀ k ∈ range N, 0 ≤ |a k| := by bound
have f4 : 0 ≤ ∑ k ∈ range N, |a k| := by apply
   sum_nonneg f3
linarith [f4]
```

1.5 Natural Language Proof

Proof: We proceed by induction on N.

Base case: If $N = 0$, then there is no natural number $n < 0$, so the statement is vacuously true.

Inductive step: Assume the result holds for some $N \geq 0$, and let $n < N + 1$.

First, observe that:

$$\sum_{k=0}^{N} |a(k)| = \sum_{k=0}^{N-1} |a(k)| + |a(N)|$$

We split into two cases:

Case 1: If $n < N$, then by the inductive hypothesis:

$$|a(n)| \leq \sum_{k=0}^{N-1} |a(k)|$$

Since $|a(N)| \geq 0$, we have:

$$|a(n)| \leq \sum_{k=0}^{N-1} |a(k)| \leq \sum_{k=0}^{N-1} |a(k)| + |a(N)| = \sum_{k=0}^{N} |a(k)|$$

Case 2: If $n \not< N$ but $n < N+1$, then $n = N$. We need to show:

$$|a(N)| \leq \sum_{k=0}^{N} |a(k)|$$

Since each $|a(k)| \geq 0$ for $k \in \{0, 1, \ldots, N-1\}$, we have $\sum_{k=0}^{N-1} |a(k)| \geq 0$. Therefore:

$$|a(N)| \leq \sum_{k=0}^{N-1} |a(k)| + |a(N)| = \sum_{k=0}^{N} |a(k)|$$

This completes the induction. **QED**

1.6 Why This Matters

This result demonstrates a fundamental principle: individual terms are controlled by aggregate sums. This is crucial when we need to bound specific terms using global information about the sequence. The theorem will be immediately applied in proving that convergent sequences are bounded.

2 Boundedness of Convergent Sequences

One of the most important properties of convergent sequences is that they must be bounded. Intuitively, this makes sense: if a sequence is approaching some finite limit, it can't wander off to infinity. The terms must stay within some fixed distance of the limit.

This theorem has profound implications. It tells us that convergence imposes strong constraints on a sequence – convergent sequences can't exhibit wild, unbounded behavior. This is used extensively in analysis to rule out certain pathological cases.

2.1 New Tools

2.1.1 Definition: SeqBdd

A sequence $a : \mathbb{N} \to \mathbb{R}$ is **bounded** (denoted `SeqBdd a`) if there exists some positive real number M such that $|a(n)| \leq M$ for all $n \in \mathbb{N}$.

Formally:

$$\text{SeqBdd}(a) \iff \exists M > 0, \forall n \in \mathbb{N}, |a(n)| \leq M$$

2.2 The Mathematical Statement

Theorem (BddOfConvNonzero): If $a : \mathbb{N} \to \mathbb{R}$ converges to L with $L \neq 0$, then a is bounded.

2.3 Strategic Approach

The proof strategy involves two parts:

1. **Eventually bounded:** Use the convergence to show that eventually (for $n \geq N$) all terms stay within distance $|L|$ of L, hence $|a(n)| \leq 2|L|$ for large n.

2. **Initially bounded:** For the finitely many terms before N, use the TermLeSum theorem to bound each by the sum $\sum_{k=0}^{N-1} |a(k)|$.

3. **Global bound:** Combine both bounds to get:

$$M = 2|L| + \sum_{k=0}^{N-1} |a(k)|$$

This bound works for all terms: the early terms are covered by the finite sum, and the later terms are covered by the $2|L|$ part.

2.4 Lean Solution

```
Statement BddOfConvNonzero (a : ℕ → ℝ) (L : ℝ) (ha :
   SeqLim a L) (hL : L ≠ 0) :
    SeqBdd a := by
choose N hN using EventuallyBdd_of_SeqConv a L ha hL
use 2 * |L| + ∑ k ∈ range N, |a k|
have absL : 0 < |L| := by apply abs_pos_of_nonzero hL
have f1 : ∀ k ∈ range N, 0 ≤ |a k| := by bound
have f2 : 0 ≤ ∑ k ∈ range N, |a k| := by apply
   sum_nonneg f1
split_ands
linarith [f2, absL]
intro n
by_cases hn : N ≤ n
specialize hN n hn
linarith [hN, f2]
have hn' : n < N := by bound
have f3 : |a n| ≤ ∑ k ∈ range N, |a k| := by apply
   TermLeSum a N n hn'
linarith [f3, absL]
```

2.5 Natural Language Proof

Proof: Since $a(n) \to L$ and $L \neq 0$, by the `EventuallyBdd_of_SeqConv` theorem, there exists N such that $|a(n)| \leq 2|L|$ for all $n \geq N$.

Define:

$$M := 2|L| + \sum_{k=0}^{N-1} |a(k)|$$

We first verify that $M > 0$. Since $L \neq 0$, we have $|L| > 0$, so $2|L| > 0$. Additionally, since each $|a(k)| \geq 0$, we have $\sum_{k=0}^{N-1} |a(k)| \geq 0$. Therefore $M > 0$.

Now we show $|a(n)| \leq M$ for all $n \in \mathbb{N}$. We consider two cases:

Case 1: If $n \geq N$, then by our choice of N:

$$|a(n)| \leq 2|L| \leq 2|L| + \sum_{k=0}^{N-1} |a(k)| = M$$

Case 2: If $n < N$, then by `TermLeSum`:

$$|a(n)| \leq \sum_{k=0}^{N-1} |a(k)| < 2|L| + \sum_{k=0}^{N-1} |a(k)| = M$$

Therefore a is bounded. **QED**

2.6 Importance of Boundedness

The boundedness of convergent sequences is fundamental in analysis. It's used to prove:

- The Bolzano-Weierstrass theorem (every bounded sequence has a convergent subsequence)
- Various compactness results
- Uniform convergence theorems
- Properties of continuous functions on compact sets

This theorem also helps develop intuition: convergence to a finite limit means the sequence can't escape to infinity. It must remain trapped in a bounded region of the real line.

In Lean, when you write $3 = succ(succ(succ(zero)))$

In set theory, $0 = \{\}$, $1 = \{\{\}\}$, $2 = \{\{\{\}\}, \{\}\}$,
$3 = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}$. $= \{0, 1\}$,

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$Add : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$.

Def defn: $n+m := \begin{cases} m=0, & n+0 = n. \\ m = succ(m'), & n+m := succ(n+m'), \end{cases}$

Thm: Given $a : \mathbb{N} \to \mathbb{R}$, $N : \mathbb{N}$

Goal: $\underline{\forall n < N}$, $|a_n| \le \sum_{\underline{k \in Range\, N}} |a_k|$

$\{0, \ldots, N-1\}$.

Note: Goal starts with $\forall n$, so shouldn't we start with intro n h_n? No! Need induction on N, not n. By induction on N

Subgoal 1: $\forall n < 0$, $|a_n| \le \sum_{k \in Range\, 0} |a_k|$

intro n h_n ($: \underline{n<0}$). contradiction.

$\boxed{P \to Q \quad h : False}$ $\underline{False \to P}$, $\neg P \to True$

$N-1$

subgoal 2: hN: $\forall n < N$, $|a_n| \le \sum_{k=0}^{N-1} |a_k|$

Goal: $\forall n < N+1$, $|a_n| \le \sum_{k=0}^{N-1+1} |a_k|$,

intro n hn.

Contrapositive: $P \to Q \Leftrightarrow \neg Q \to \neg P$.

have f1: $\sum_{k \in \text{range } N+1} |a_k| = \sum_{k \in \text{range} N} |a_k| + |a_N|$, := by apply sum_range_succ

rewrite [f1]

have f2: $\forall k \in \text{range} N$, $0 \le |a_k|$:= by bound.

have f3: $0 \le \sum_{k \in \text{range} N} |a_k|$:= by apply sum_nonneg. f2.

have f4: $0 \le |a_N|$:= by bound.

Goal: $|a_n| \le \sum_{k \in \text{range} N} |a_k| + |a_N|$,

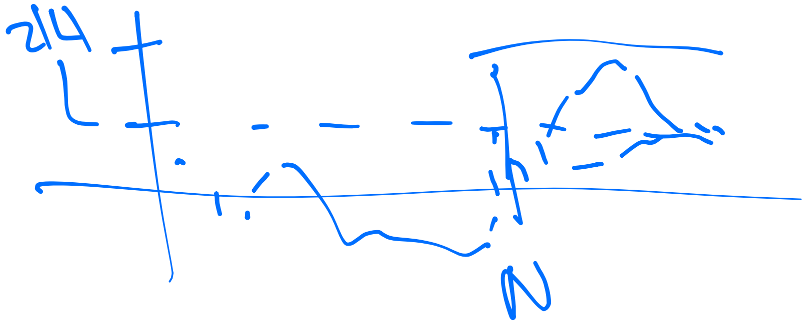
by_cases h: $n < N$.

1. specialize hN n h, hN: $|a_n| \le \sum_{k \in \text{range} N} |a_k|$.
   - linarith [f4, hN].
2. h: $\neg\, n < N$.
   - have f5: $n = N$:= by bound.
   - rewrite [f5], Goal: $|a_N| \le \sum^{N} |a_k| + |a_N|$
   - linarith [f3].

Ex: $(ha : \forall n,\ a\, n = n) \Rightarrow \neg$ SeqBdd a.

Thm: If $a_n \to L$ $(hL : L \neq 0)$, then $\{a_n\}$ is bounded

def SeqBdd $(a : \mathbb{N} \to \mathbb{R})$: Prop := $\exists M > 0,$
$\forall n, |a_n| \le M.$



Proof:

$\boxed{\text{EventuallyBddOfSeqConv } a\ L\ (ha : a \to L)\ (L \neq 0) : \exists N, \forall n \ge N, |a_n| \le 2|L|.}$

Idea: after $n \ge N$, use $2 \cdot |L|$.
before: add them!. Total upper bound: $2 \cdot |L| + \sum_{k \in \text{range } N} |a_k|$

Goal: $\exists M > 0, \forall n, |a_n| \le M.$

~~have h0 : $\exists N, \forall n \ge N, |a_n| \le 2 \cdot |L|$:= by apply~~ EventBddOfSeqConv a L ...

choose N hN

Use $2 \cdot |L| + \sum |a_k|$. Goal: $2|L| + \sum > 0 \land \forall n, |a_n| \le 2|L| + \sum$.

have f1: $0 \le \sum |a_k|$ $k \in$ range N
have f2: $0 < |L|$:= by apply sum_nonneg (by bound).
split_ands := by apply abs_pos_of_nonzero hL.

↳ SubGoal: $0 < 2|L| + \sum$.
linarith[f1, f2]

↳ SubGoal: $\forall n, |a_n| \le 2|L| + \sum^{N} |a_k|$.

intro n

by_cases h : n < N

↳ have f3 : $|a_n| \le \sum^N |a_k|$:= by

apply Trans_sum.

linarith [f3, f2].

↳ h : ¬ n < N

have h' : N ≤ n := by dong

specialize hN n h'

linarith [hN, f1].

Current Goal State:

$a : \mathbb{N} \to \mathbb{R}$

$L : \mathbb{R}$ ha : Seqlim a L.

hL : $L \neq 0$.

N n : $\mathbb{N}$

hN : $\forall n \geq N,\ |a_n| \leq 2|L|$.

$f_1 : 0 \leq \sum |a_k|$

$f_2 : 0 < |L|$

Goal: $|a_n| \leq 2|L| + \sum |a_n|$.

Thm (Prod Lim) $a_n \to L \neq 0$, $b_n \to M \neq 0$, $c_n = a_n \cdot b_n$.

Goal: $c_n \to L \cdot M$.

intro ε hε

Key: Need $|a_n \cdot b_n - L \cdot M| < \varepsilon$.

Idea: $\left|(a_n \cdot b_n - b_n \cdot L) + (b_n \cdot L - L \cdot M)\right|.$

$$\le \underbrace{|b_n|}_{<K} \cdot \underbrace{|a_n - L|}_{<\varepsilon_1 = \frac{\varepsilon}{2K}} + \underbrace{|L|}_{> 0} \cdot \underbrace{|b_n - M|}_{<\varepsilon_2 = \frac{\varepsilon}{2|L|}}.$$