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Lecture 5: Doubling a Convergent Sequence

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1 The Factory Scaling Challenge

After conquering the constant sequence, let's up our game: if a sequence converges, then doubling that sequence also converges, and converges to double the original limit.

Imagine our Machinist receives a challenge from the Engineer: 'Please double all the lengths, but maintain the same quality standards.' How should the Machinist respond?

If the Engineer demands the doubled lengths be within ε of $2L$, the Machinist can't just demand that the original process meet the original tolerance ε, because

$$|2 \cdot a(n) - 2 \cdot L| < 2 \cdot \varepsilon,$$

not ε. Instead, we must be more clever. Can you think of what to do?

That's right, the ε in the original process is *arbitrary*, so we can play with it! If we could get the original process to guarantee output within $\varepsilon/2$ of L, then doubling that output will be within ε of $2L$.

This is the key insight: **when scaling by a constant, we need to scale our tolerance demands inversely**.

2 The Mathematical Setup

Suppose we have:

- A sequence $a : \mathbb{N} \to \mathbb{R}$ that converges to L
- A sequence $b : \mathbb{N} \to \mathbb{R}$ with the property that $b(n) = 2 \cdot a(n)$ for all n

We want to prove that b converges to $2 \cdot L$.

2.1 Key Insight: Inverse Tolerance Scaling

The crucial observation is that:

$$|b(n) - 2L| = |2 \cdot a(n) - 2L| = 2 \cdot |a(n) - L|$$

So if we want $|b(n) - 2L| < \varepsilon$, we need $2 \cdot |a(n) - L| < \varepsilon$, which means $|a(n) - L| < \varepsilon/2$.

This is exactly what we can get from the convergence of a!

3 New Tools You'll Need

3.1 Absolute Value of Products

You'll need the new theorem `abs_mul` which states that for any real numbers x and y:

$$|x \cdot y| = |x| \cdot |y|.$$

To use this theorem, you may find it convenient to make a new hypothesis using `have` and then `rewrite` by that hypothesis. That is, you can say,

```
have NewFact : |Something * SomethingElse| =
|Something| * |SomethingElse| :=
by apply abs_mul
```

and then `rewrite [NewFact]` will replace `|Something * SomethingElse|` by `|Something| * |SomethingElse|` (either at the Goal, or `at` a hypothesis, if you so specify).

4 Strategic Approach

1. Unfold the definition of convergence for the goal
2. Given any $\varepsilon > 0$, use the convergence of a with tolerance $\varepsilon/2$. You may find it useful to separately prove the inequality $0 < \varepsilon/2$ – which tactic do you think will help with that?
3. Extract the witness N from the convergence of a
4. Use the same N for your sequence b
5. Apply the scaling hypothesis and use `abs_mul` to relate $|b(n) - 2L|$ to $|a(n) - L|$
6. Use the convergence bound for a to conclude

5 Lean Solution

```
Statement (a b : ℕ → ℝ) (L : ℝ)
    (h : SeqLim a L) (b_scaled : ∀ n, b n = 2 * a n) :
    SeqLim b (2 * L) := by
  change ∀ ε > 0, ∃ N : ℕ, ∀ n ≥ N, |b n - 2 * L| < ε
  intro ε hε
  change ∀ ε₁ > 0, ∃ N₁ : ℕ, ∀ n ≥ N₁, |a n - L| < ε₁ at
      h
  specialize h (ε / 2)
  have eps_half_pos : 0 < ε / 2 := by bound
  specialize h eps_half_pos
  choose N hN using h
  use N
  intro n hn
  specialize b_scaled n
  rewrite [b_scaled]
  have factor : 2 * a n - 2 * L = 2 * (a n - L) := by
     ring_nf
  rewrite [factor]
  have abs_factor : |2 * (a n - L)| = |2| * |a n - L| :=
      by apply abs_mul
  rewrite [abs_factor]
```

```
specialize hN n hn
norm_num
bound
```

6 Natural Language Proof

Let's step back from the formal Lean proof and understand what we just proved in plain English.

6.1 Theorem Statement

Theorem: If a sequence of real numbers converges to some limit, then the sequence formed by doubling each term converges to double the original limit.

6.2 Proof

Proof: Suppose sequence a_n converges to L, and we want to show that $b_n = 2 \cdot a_n$ converges to $2L$.

By definition, we need to show that for any tolerance $\varepsilon > 0$, we can find a point N such that for all $n \geq N$, we have $|b_n - 2L| < \varepsilon$.

Here's the key insight: Since a_n converges to L, we can make $|a_n - L|$ arbitrarily small. Specifically, we can find an N such that $|a_n - L| < \varepsilon/2$ for all $n \geq N$.

Now, for any $n \geq N$:

$$
\begin{aligned}
|b_n - 2L| &= |2a_n - 2L| && (1)\\
&= |2(a_n - L)| && (2)\\
&= 2|a_n - L| && (3)\\
&< 2 \cdot \frac{\varepsilon}{2} = \varepsilon && (4)
\end{aligned}
$$

Therefore, b_n converges to $2L$, completing the proof. **QED**

7 Why This Pattern Matters

This proof introduces the important technique of **inverse scaling** for tolerances. When you scale a sequence by a constant c, you need to scale your tolerance demands by $1/c$. This principle will appear again when you study:

- Products of sequences (where both factors contribute to the error)
- Derivatives (where the limit definition involves scaling by h)
- Integration (where Riemann sums involve scaling by partition widths)

The ability to manage how constants affect convergence is fundamental to all of analysis!

7.1 The Elegant Strategy

Your proof used the **inverse scaling** principle:

1. **Tolerance inversion**: To achieve tolerance ε for doubled output, demand tolerance $\varepsilon/2$ for original input
2. **Algebraic factoring**: $2 \cdot a_n - 2 \cdot L = 2 \cdot (a_n - L)$ revealed the structure
3. **Absolute value scaling**: $|2 \cdot x| = |2| \cdot |x|$, followed by `norm_num`, converted the factored form to the needed bound
4. **Linear arithmetic**: The final `bound` combined $2 \cdot |a_n - L| < 2 \cdot (\varepsilon/2) = \varepsilon$