An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 25: Swapping Limits and Integrals

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SIMPLICIO: We meet one last time, Socrates.

SOCRATES: For now, old friend.

SIMPLICIO: Hey, quick question. Why do we need general indexing types, can't we just index everything by natural numbers?

SOCRATES: Ah, great question! Besides the generality that abstract indexing types provide, there's a crucially important reason: sometimes, the objects we want to index by have *no* possible indexing by natural numbers at all!

SIMPLICIO: Really? Like what?

SOCRATES: Let's back up a little. Professor David Hilbert liked to tell this story in terms of a magical hotel. So: imagine a hotel with infinitely many rooms, numbered $R_1, R_2, R_3, \ldots$. Now, suppose that every room is occupied, with guest G_1 in room R_1, guest G_2 in room R_2, and so on. There are infinitely many rooms, but also infinitely many guests, so everybody has a room, and every room is taken. $\infty = \infty$. So far so good?

SIMPLICIO: I mean, yeah, except for the infinitely large hotel and infinitely many people thing...

SOCRATES: Come on, go with me here. Now, on the first night, a new guest, G_0, arrives, asking for a place to stay. The clerk says, "Hey buddy, can't you read? No vacancy. Sorry!"

Professor Hilbert overhears this from his office, and comes around to the front desk. He says, "Wait a minute, sir; I'm sorry, yes, we can host you tonight."

The clerk protests: "But Professor, all the rooms are taken! Just because the hotel has infinitely many rooms doesn't mean you can ask G_0 to go all the way to the 'end' somehow, right?"

Hilbert replies: "You're absolutely right, that would be impossible. I can't tell guest G_0 to go to 'the end', because that's not a room assignment. Here's the hotel's ledger; every guest needs to have exactly one room assigned to them. But! That doesn't necessarily mean that we're out of luck," and he gets on the hotel loudspeaker: "Attention all guests! I'm sorry to inconvenience you, but may I please request that the guest in room R_n move to room R_{n+1}? Thank you!"

Hilbert turns to the guest, "In just a moment, sir, room R_1 will be free for you." He asks the clerk to update the ledger, and returns to his office.

SIMPLICIO: Wow, that's clever! So wait, $\infty + 1 = \infty$ after all?

SOCRATES: You ain't seen nothin' yet, kid. On the second night, the hotel is again full, with guest G_1 back to room R_1, and so on.

SIMPLICIO: Oh so guest G_0 left, and everybody had to move again? I don't think I'd like to stay at this hotel...

SOCRATES: Shush. On the second night, a car pulls up with 10 people in it. The clerk knows exactly what to do; he gets on the speaker and says, "Will the guest in room R_n please move to room R_{n+10}? Thank you!" And just like that, rooms R_1 through R_{10} are free for the new guests.

Then on the third night, a bus arrives, but this time it has *infinitely* many passengers, P_1, P_2, P_3, and so on. The clerk says to them, "Gee, I'm really sorry. I could accommodate 100 or 1,000 or even 1,000,000 of you, but I can't tell the guest in room R_n to move to room $R_{n+\infty}$, that's not a room assignment! I'm really sorry, but we just won't be able to host you tonight."

SIMPLICIO: Well, yeah, now you're asking for $\infty+\infty$! Surely that's bigger than ∞... No?

SOCRATES: Professor Hilbert again overhears this, and comes out from his office. He says to the bus driver, "Please give me just a moment, and we'll make space for you all."

The clerk protests, "But Professor, how can we do that? You said I couldn't send anyone infinitely far down the hotel?"

Hilbert, ignoring the clerk, gets on the speaker: "Attention all guests! I'm sorry to inconvenience you, but may I please request that the guest in room R_n move to room R_{2n}? Thank you!" Hilbert turns to the clerk: "And now, all the odd numbered rooms are free for the bus passengers. Please update the ledger accordingly."

SIMPLICIO: Whoa, cool! So $\infty + \infty = \infty$ too!

SOCRATES: On the fourth night, the bus has left, and the original guests are back in their rooms. Now ten buses arrive, each with infinitely many passengers. The clerk knows exactly what to do.

SIMPLICIO: Ok, I get it. He gets on the speaker and tells the guest in room R_n to move to room R_{10n}, right? Then passengers from bus 1 go to rooms $R_1, R_{11}, R_{21}, \ldots$, the passengers from bus 2 go to rooms $R_2, R_{12}, R_{22}, \ldots$, and so on. That's cool, so $10 \times \infty = \infty$ too.

SOCRATES: Right! On the fifth night, the hotel is back as it started, and a caravan arrives of *infinitely many* buses, each with *infinitely many* passengers. Bus 1 has passengers $P_{1,1}, P_{1,2}, P_{1,3}, \ldots$, and bus 2 has passengers $P_{2,1}, P_{2,2}, P_{2,3}, \ldots$, and so on.

The clerk says to them, "Look, I think we don't have room, because I can't tell the guest in room R_n to move to room $R_{n\times\infty}$. But this Professor here is a real whiz; he just might have something up his sleeve."

Hilbert comes out from his office, and says to the clerk, "Indeed, we can handle them all! Please ask the guests to move to twice their room number as before, to free up infinitely many rooms."

The clerk starts to protest, thinking the absentminded Professor might have not seen that the caravan has infinitely many buses. But Hilbert cuts him off: "Here's what we're going to do: In the first available room, please place passenger $P_{1,1}$. Then $P_{1,2}$, then $P_{2,1}$. Then $P_{3,1}$, $P_{2,2}$, and $P_{1,3}$, and so on, zigzagging through the passenger list. As you'll see, everyone has a clearly defined, finite room number, with nobody left behind."

SIMPLICIO: Whoa, that's crazy, even $\infty \times \infty = \infty$! So I guess infinity is so big, you can fit just about anything!

SOCRATES: Well, now it's the sixth night, and the hotel is empty. I guess the guests got tired of being moved around so much, and the reviews started to tank! Only one bus arrives. But this is an extremely large party bus from Sweden, blasting ABBA tunes. Everybody on this bus has a very strange name. It consists of an infinite sequence of As and Bs. For example, one passenger has the name $ABBAAABABBA\ldots$, going on forever. And another is called $AAABBBBBAA\ldots$, and so on. In fact, for *any* sequence of As and Bs, there is a single passenger on this bus with exactly that name. Their manager comes off the bus to speak with the clerk.

The clerk says, "Gee whiz, that's a pretty big bus, sir. But don't worry, I'm *sure* we can find a way to fit everybody. Let me just get the Professor."

Hilbert steps out of his office, and sighs with disappointment. "I'm very sorry," he says to the manager, "but we will *not* be able to accommodate your passengers."

The clerk is shocked. "But Professor, why not? We've handled infinitely many buses with infinitely many passengers before! This is just *one* bus!"

Hilbert says to the clerk, "Take out the ledger. Let's work backwards. Imagine that we're done placing *every single person* from that bus into a room. So in room R_1, we have $ABBAAA\ldots$, say, and in room R_2, we have $AAABBB\ldots$, and in room R_3 is $BABBAA\ldots$ and so on. Ok? I claim we're not done, and we've left someone off the ledger. If I can prove to you that at least *one* passenger hasn't been assigned a room, then do you see how the entire enterprise is futile?"

The clerk nods reluctantly. Hilbert continues: "The person in room R_1 is named $ABBAAA\ldots$, which starts with A. So if I give you the name of a person whose first letter is B, then that person is certainly not in room R_1, right?" The clerk nods. "Now, the person in room R_2 is named $AAABBB\ldots$. If I give you the name of a person whose *second* letter is B, then that differs from the second letter of the person in room R_2, and so that person is not in room R_2. Do you follow?" The clerk nods again.

SIMPLICIO: Oh, I see it!! You're going to take the name of the person in room R_n, and change the n-th letter! That way, you'll make a name that differs from *everybody's* name in the ledger, by at least one letter! So whoever that passenger is, and it's certainly someone on the bus, they can't be in any room! Whoa! So wait, how big of an infinity is on this bus?

SOCRATES: Indeed! Let's think. If we only had names that were four letters long, like $ABBA$ or $AABB$ or $BABA$, how many such names are there?

SIMPLICIO: Easy peasy, there are $2^4 = 16$ such names, since each of the four letters can be either A or B. Oh! So the ABBA bus has 2^∞ people on it? Whoa, and we just showed that $2^\infty > \infty$!

SOCRATES: Exactly right. This is called the *Cantor diagonalization argument*, and it shows that there are different sizes of infinity. So let me ask you: what the heck does any of this have to do with indexing sets?

SIMPLICIO: Ok, I see it! The first night we just reindexed the naturals $\mathbb{N}$ by $n \mapsto n+1$. The third night, we indexed all the integers $\mathbb{Z}$, which are 'twice' as large as $\mathbb{N}$. That is, we can list every element of $\mathbb{Z}$ as: 0, -1, 1, -2, 2, and so on (the even indices are the original sequence, and the odd indices are the negative newcomers!). What's the meaning of the fifth night?

SOCRATES: So then we had infinitely many buses with infinitely many passengers. We can think of the buses as denominators, and passengers as numerators. We indexed the rationals, $\mathbb{Q}$! We can assign an ordering to the rationals by zigzagging through the fraction table, just like we did with the infinitely many passengers on infinitely many buses.

SIMPLICIO: Hah! So cool! And the sixth night?

SOCRATES: Think about the real numbers $\mathbb{R}$. Can we index those, assign *every* real number to a room in the hotel? I claim we can't even do it for all the reals in $[0, 1]$. In fact, just look at the decimals in that range that use only 0s and 1s, like $0.101100111\ldots$ and $0.111000101\ldots$. That *is* the ABBA bus, and those real numbers can't be indexed by $\mathbb{N}$, so certainly all of the reals can't either!

`https://youtu.be/OxGsU8oIWjY`

SIMPLICIO: That's wild stuff! Hey, if 2^∞ is strictly bigger than ∞, in this sense, is there some infinity, like, in between them?

SOCRATES: Whoa, hold your horses, lad! That question is too deep for us now. (Go google the 'Continuum Hypothesis'. It turns out: that depends on what the definition of "is" is... Don't google *that*.) All I want you to understand for now is that there are surprisingly many things that we *can* index by $\mathbb{N}$ (by the way, anything that can be indexed by $\mathbb{N}$ is called *countable*), but there are also many "*uncountable*" things, that is, those that cannot be indexed by $\mathbb{N}$. So we need general indexing types to handle all the different situations that arise in analysis.

SIMPLICIO: Cool! So what's today's lesson?

SOCRATES: Well, I still owe you a justification for Newton's term-by-term integration of infinite series. So let's jump straight there. If you recall, Newton argued that, since

$$\sqrt{1-x^2} = 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} - \cdots,$$

he could integrate term by term:

$$\int_0^{1/2} \sqrt{1-x^2}\, dx = \int_0^{1/2} 1dx - \int_0^{1/2} \frac{x^2}{2} dx - \int_0^{1/2} \frac{x^4}{8} dx - \int_0^{1/2} \frac{x^6}{16} dx - \cdots.$$

In other words, he had a sequence of functions $f_n(x)$ (the sum of the first n terms of the series), and he claimed that since f_n converged to $F(x) = \sqrt{1-x^2}$, then

$$\int F = \lim(\int f_n),$$

or

$$\int (\lim f_n) = \lim(\int f_n).$$

So is this actually true, and under what conditions can it be justified?

SIMPLICIO: Ok, let me guess. Of course each f_n had better be integrable, otherwise the whole thing falls apart. But I bet you're going to tell me a slew of examples where pointwise convergence isn't enough?

SOCRATES: You know me very well by now. Indeed, you need to keep just a few examples in mind, and you'll see clearly what's going on, and what can go wrong. Let's start with this example. Let $f_n(x)$ be the indicator function of the interval $[n, n+1]$. That is, it's 1 if $x \in [n, n+1]$, and 0 otherwise. Each f_n is integrable, since it's just a step function, and has integral 1. So $\lim(\int f_n) = 1$.

But for any fixed x, what happens to the limit of $f_n(x)$?

SIMPLICIO: Yep, I see it, eventually $f_n(x)$ becomes 0 for all sufficiently large n, once n exceeds x. So the pointwise limit function $F(x)$ is identically 0, which of course has integral $\int F = 0 \neq 1$. What else? Maybe this example is bad because you're shifting the intervals off to infinity? What if we keep all the action in $[0, 1]$?

SOCRATES: Good idea. Let's try this: let $f_n(x) = n$ if $x \in [0, 1/n]$, and 0 otherwise, and let's integrate it on the nice, *compact* region $[0, 1]$. What's going on there?

SIMPLICIO: Ok, each f_n is integrable, since it's again a step function. The integral is $\int f_n = n \times (1/n) = 1$. So again $\lim(\int f_n) = 1$. But for any fixed $x > 0$, eventually n gets so large that x is outside $[0, 1/n]$ (the Archimedean property!!), so $f_n(x) = 0$ for all sufficiently large n. And at $x = 0$, well, $f_n(0) = n$, which diverges. So the pointwise limit function $F(x)$ is again identically 0 for all $x > 0$, and undefined at $x = 0$. But I'm guessing a single point doesn't affect the integral; we can just say $F(x) = 0$ everywhere, and so again $\int F = 0 \neq 1$. Ah, but these functions are not continuous! Maybe that's the problem?

SOCRATES: Good thinking. Let's try to make them continuous. Consider the functions $f_n(x)$ defined as follows:

- $f_n(x) = n^2 x$ for $x \in [0, 1/n]$
- $f_n(x) = -n^2 x + 2n$ for $x \in [1/n, 2/n]$
- $f_n(x) = 0$ for $x \notin [0, 2/n]$

So we've made a little "tent" shape that peaks at height n at $x = 1/n$, and goes back down to 0 by $x = 2/n$. Each f_n is continuous, and integrable, with integral $\int f_n = 1$, because the area is a triangle with base $2/n$ and height n. So $\lim(\int f_n) = 1$. But again, for any fixed $x > 0$, eventually n gets so large that x is outside $[0, 2/n]$, so $f_n(x) = 0$ for all sufficiently large n. And at $x = 0$, we have $f_n(0) = 0$ for all n. So the pointwise limit function $F(x)$ is again identically 0 for all x, and so again $\int F = 0 \neq 1$. Hmm, so continuity isn't enough either.

SIMPLICIO: Wow, these are some tricky examples. Ok, so what is it, what's the condition we need to make this work?

SOCRATES: You already know it!

SIMPLICIO: Oh... Is it... *uniform convergence*? Let's think. Suppose f_n is integrable, and converges to F uniformly. Let's think about the difference between the order N Riemann sum for F, and the order N Riemann sum for f_n. Since the convergence is uniform, for any $\varepsilon > 0$, there exists M such that for all $n \geq M$, we have $|f_n(x) - F(x)| < \varepsilon$, regardless of what x is. So the difference between the Riemann sums will be:

$$\left| \left(\sum_{i=1}^{N} f_n(x_i) \Delta x \right) - \left(\sum_{i=1}^{N} F(x_i) \Delta x \right) \right| \leq \sum_{i=1}^{N} |f_n(x_i) - F(x_i)| \Delta x$$

$$< \sum_{i=1}^{N} \varepsilon \Delta x \ = \ N \times \varepsilon \times \frac{b-a}{N} \ = \ \varepsilon(b-a).$$

Cool! That means that we can *simultaneously* show that F is integrable (by comparing its Riemann sums to those of f_n for large n), and that the integrals of the f_ns converge to the integral of F (by the same reasoning). So uniform convergence should do it all for us!

SOCRATES: Exactly right! So let's finish strong with a proof!

Level 1: Uniform Convergence Implies Integrability

In this level, we prove the fundamental theorem that justifies Newton's term-by-term integration of infinite series: if a sequence of integrable functions converges uniformly to a function, then that function is also integrable, and its integral equals the limit of the integrals.

The Theorem

Theorem (`Integrable_of_UnifConv`): If f_n converges uniformly to F, and each f_n is integrable on $[a, b]$, then F is integrable on $[a, b]$, and

$$\int_a^b F\,dx = \lim_{n\to\infty} \int_a^b f_n\,dx$$

In other words: *uniform convergence preserves integrability and allows interchange of limits and integrals.*

Proof Strategy

The proof has three main parts:

1. **Riemann Sum Convergence**: Show that for any Riemann sum approximation, the difference between the Riemann sums of f_n and F can be made arbitrarily small uniformly in n.

2. **Cauchy Convergence of Integrals**: Use the uniform convergence to show that the sequence of integrals $\int_a^b f_n\,dx$ is Cauchy and hence converges to some limit L.

3. **Integrability of the Limit**: Show that F is integrable with integral equal to L by using the uniform bounds on Riemann sum differences.

Key Insight

The crucial observation is that uniform convergence gives us control over all function values simultaneously. If $|f_n(x) - F(x)| < \varepsilon/(b-a)$ for all $x \in [a, b]$ and sufficiently large n, then for any Riemann sum with partition points x_i and mesh size Δx:

$$\left|\sum_i f_n(x_i)\Delta x - \sum_i F(x_i)\Delta x\right| \leq \sum_i |f_n(x_i) - F(x_i)|\Delta x < \sum_i \frac{\varepsilon}{b-a}\Delta x = \varepsilon$$

This uniform bound on Riemann sum differences is the key to both proving convergence of the integrals and integrability of the limit function.

The Formal Proof

Proof: Let $\varepsilon > 0$ be given.

Step 1: By uniform convergence, choose N such that for all $n \geq N$ and all $x \in [a, b]$:

$$|f_n(x) - F(x)| < \frac{\varepsilon}{b-a}$$

This gives us uniform control over the difference between f_n and F.

Step 2: For any Riemann sum approximation with M intervals, we have:

$$|RiemannSum(f_n, a, b, M) - RiemannSum(F, a, b, M)| < \varepsilon$$

This follows from the triangle inequality applied to the Riemann sum differences.

Step 3: We show the sequence $\{\int_a^b f_n\,dx\}$ is Cauchy. For $n, m \geq N$ and any Riemann sum parameter M:

$$\left|\int_a^b f_n\,dx - \int_a^b f_m\,dx\right| \leq \text{(Riemann sum approximation errors)} + \text{(uniform convergence bound)}$$

By choosing M large enough to make the Riemann sum approximation errors small, and using our uniform bound, we can make this difference arbitrarily small.

Step 4: By the completeness of $\mathbb{R}$, the sequence of integrals converges to some limit L.

Step 5: We show F is integrable with integral L. For any $\varepsilon > 0$ and sufficiently large n and M:

$$|RiemannSum(F, a, b, M) - L| \leq |RiemannSum(F, a, b, M) - RiemannSum(f_n, a, b, M)|$$
$$+ |RiemannSum(f_n, a, b, M) - \int_a^b f_n\,dx| + \left|\int_a^b f_n\,dx - L\right|$$

Each term can be made small by our choices of n and M, completing the proof. □

The Lean Proof

```
Statement Integrable_of_UnifConv {f : ℕ → ℝ → ℝ} {F : ℝ
  → ℝ}
    {a b : ℝ} (hab : a < b)
    {ℓ : ℕ → ℝ}
    (hfint : ∀ n, HasIntegral (f n) a b (ℓ n))
    (hfF : UnifConv f F) :
    ∃ (L : ℝ), SeqLim ℓ L ∧ HasIntegral F a b L
 := by
have RSdiff : ∀ ε > 0, ∃ N, ∀ n ≥ N, ∀ M, |RiemannSum (f
    n) a b M - RiemannSum F a b M| < ε := by
  intro ε hε
  choose N hN using hfF (ε / (b - a)) (by bound)
  use N
  intro n hn M
  specialize hN n hn
   sorry
have Lconv : SeqConv ℓ := by
  apply SeqConv_of_IsCauchy
  intro ε hε
  choose N hN using RSdiff (ε / 4) (by bound)
  use N
  intro n hn m hmn
  choose M1 hM1 using hfint n (ε / 4) (by bound)
  choose M2 hM2 using hfint m (ε / 4) (by bound)
  let M := M1 + M2
  specialize hM1 M (by bound)
  specialize hM2 M (by bound)
  have hNn := hN n hn M
  have hNm := hN m (by bound) M
  have b1 : |ℓ m - ℓ n| ≤ |RiemannSum (f n) a b M - ℓ n|
      + |RiemannSum (f n) a b M - RiemannSum F a b M| +
    |RiemannSum (f m) a b M - RiemannSum F a b M| + |
      RiemannSum (f m) a b M - ℓ m| := by
    sorry
  linarith [b1, hM1, hM2, hNn, hNm]
choose L hL using Lconv
use L, hL
intro ε hε
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choose N1 hN1 using hL (ε / 3) (by bound)
choose N2 hN2 using RSdiff (ε / 3) (by bound)
let N := N1 + N2
have Nbnd1 : N1 ≤ N := by bound
have Nbnd2 : N2 ≤ N := by bound
specialize hN1 N (by bound)
specialize hN2 N (by bound)
choose Mmax hMmax using hfint N (ε / 3) (by bound)
use Mmax
intro M hM
specialize hMmax M hM
specialize hN2 M
have b2 : |RiemannSum F a b M - L| ≤ |RiemannSum (f N) a
    b M - RiemannSum F a b M| +
  |RiemannSum (f N) a b M - ℓ N| + |ℓ N - L| := by
  sorry
linarith [b2, hMmax, hN1, hN2, Nbnd1, Nbnd2]
```

Applications

This theorem is fundamental to the rigorous foundation of calculus. Some key applications:

- **Power Series Integration**: If a power series converges uniformly on an interval, it can be integrated term by term
- **Newton's Method**: Justifies Newton's term-by-term integration in his calculation of π
- **Fourier Analysis**: Enables term-by-term integration of uniformly convergent Fourier series
- **Differential Equations**: Allows integration of series solutions when they converge uniformly

The key requirement is *uniform* convergence. As we saw in the dialogue, pointwise convergence alone is insufficient, even for continuous functions!

Level 2: The Intermediate Value Theorem

We conclude our course with one of the most fundamental and intuitive theorems in all of mathematics: the Intermediate Value Theorem. This theorem captures the essential "continuity" of the real line and has been assumed obvious by mathematicians for centuries, yet requires all the sophisticated machinery we've built to prove rigorously.

The Theorem

Theorem (`IVT`): If a function f is continuous on a closed interval $[a, b]$, and takes values $f(a) < 0$ and $0 < f(b)$, then there exists some $c \in (a, b)$ such that $f(c) = 0$.

In other words: *a continuous function that changes sign must cross zero.*

Historical Context

The Intermediate Value Theorem seems so "obvious" that it was used implicitly by mathematicians for over 2000 years before anyone thought to prove it rigorously. The ancient Greeks used it in their geometric constructions, and even Euler and his contemporaries treated it as self-evident.

The first rigorous proof was given by Bernard Bolzano in 1817, and later independently by Augustin-Louis Cauchy. Their proofs required the completeness of the real numbers—a property that wasn't even properly understood until the late 19th century!

Proof Strategy

We will use the **Least Upper Bound Principle**. The idea is to consider the set S of all points $x \in [a, b]$ where $f(x) < 0$. This set:

- Is non-empty (since $a \in S$)
- Is bounded above (by b)
- Therefore has a least upper bound c by the Completeness Axiom

We then show that $f(c) = 0$ by proving that both $f(c) < 0$ and $f(c) > 0$ lead to contradictions using the continuity of f.

The Formal Proof

Proof:

Step 1: Define $S = \{x \in [a, b] : f(x) < 0\}$.

Step 2: Show S is non-empty. Since $f(a) < 0$, we have $a \in S$.

Step 3: Show S is bounded above. For any $x \in S$, we have $x \in [a, b]$, so $x \leq b$.

Step 4: By the Least Upper Bound Principle, S has a supremum. Let $c = \sup S$.

Step 5: Show $a \leq c \leq b$. Since $a \in S$, we have $a \leq c$. Since b is an upper bound for S, we have $c \leq b$.

Step 6: Show $f(c) \not< 0$ (proof by contradiction).

Suppose $f(c) < 0$. By continuity of f at c, choose $\delta > 0$ such that for all x with $|x - c| < \delta$:

$$|f(x) - f(c)| < \frac{-f(c)}{2}$$

This implies $f(x) < \frac{f(c)}{2} < 0$ for all $x \in (c - \delta, c + \delta)$.

In particular, if $c + \delta/2 \leq b$, then $f(c + \delta/2) < 0$, so $c + \delta/2 \in S$. But this contradicts c being an upper bound for S.

If $c + \delta/2 > b$, then $f(b) < 0$, contradicting our hypothesis that $f(b) > 0$.

Step 7: Show $f(c) \not> 0$ (proof by contradiction).

Suppose $f(c) > 0$. By continuity of f at c, choose $\delta > 0$ such that for all x with $|x - c| < \delta$:

$$|f(x) - f(c)| < \frac{f(c)}{2}$$

This implies $f(x) > \frac{f(c)}{2} > 0$ for all $x \in (c - \delta, c + \delta)$.

In particular, if $c - \delta/2 \geq a$, then $f(c - \delta/2) > 0$, which means $c - \delta/2$ is an upper bound for S that's smaller than c. This contradicts $c = \sup S$.

If $c - \delta/2 < a$, then $f(a) > 0$, contradicting our hypothesis that $f(a) < 0$.

Step 8: Since $f(c) \not< 0$ and $f(c) \not> 0$, we must have $f(c) = 0$.

Step 9: Show $c \in (a, b)$. Since $f(a) < 0 = f(c)$, we have $a \neq c$. Since $f(c) = 0 < f(b)$, we have $c \neq b$. Combined with $a \leq c \leq b$, this gives $c \in (a, b)$.

Therefore, we have found $c \in (a, b)$ such that $f(c) = 0$. □

The Lean Proof

```
Statement IVT {f : ℝ → ℝ} (hf : FunCont f) {a b : ℝ} (
   hab : a < b)
    (hfa : f a < 0) (hfb : 0 < f b): ∃ c ∈ Ioo a b, f c
       = 0 := by
let S := { x ∈ Icc a b | f x < 0 }
have a_in_S : a ∈ S := by
  split_ands
  · bound
  · bound
  · apply hfa
have Snonempty : S.Nonempty := by
  use a
have Sbounded : IsUB S b := by
  intro x hx
  apply hx.1.2
choose c hc using HasLUB_of_BddNonempty Snonempty
   Sbounded
have a_le_c : a ≤ c := by
  apply hc.1 a a_in_S
have c_le_b : c ≤ b := by
  apply hc.2
  intro s hs
  apply hs.1.2
have fc_lt : ¬ f c < 0 := by
  intro h
  specialize hf c (-f c / 2) (by bound)
  choose δ hδpos hδ using hf
  have cpd : c + δ / 2 ≤ b := by
    by_contra hb
    push_neg at hb
    have hbc : |b - c| < δ := by
      rewrite [abs_lt]
      split_ands
      linarith [c_le_b, hδpos]
      linarith [hb, hδpos]
    specialize hδ b hbc
    rewrite [abs_lt] at hδ
    linarith [hδ, h, hfb]
```

```
  specialize hδ (c + δ / 2) (by ring_nf; rewrite [
     abs_of_nonneg (by bound)]; linarith [hδpos])
  rewrite [abs_lt] at hδ
  have hfc1 : f (c + δ / 2) < 0 := by
    linarith [hδ]
  have hc_in_S : c + δ / 2 ∈ S := by
    split_ands
    · bound
    · bound
    · apply hfc1
  have hc_ineq := hc.1 (c + δ / 2) hc_in_S
  linarith [hc_ineq, hδpos]
have fc_gt : ¬ 0 < f c := by
  intro h
  specialize hf c (f c / 2) (by bound)
  choose δ hδpos hδ using hf
  have cpd : a ≤ c - δ / 2 := by
    by_contra ha
    push_neg at ha
    have hac : |a - c| < δ := by
      rewrite [abs_lt]
      split_ands
      bound
      bound
    specialize hδ a hac
    rewrite [abs_lt] at hδ
    linarith [hδ, hfa, h]
  have cUB : IsUB S (c - δ / 2) := by
    intro s hs
    by_contra hsc
    push_neg at hsc
    have s_le : s ≤ c := by
      apply hc.1 s hs
    have hcs : |s - c| < δ := by
      rewrite [abs_lt]
      split_ands
      bound
      bound
    specialize hδ s hcs
    have hfs : f s < 0 := by
```

```
      apply hs.2
    rewrite [abs_lt] at hδ
    linarith [hδ, hfs, h]
  linarith [hc.2 (c - δ / 2) cUB, hδpos]
have fc : f c = 0 := by bound
have hc' : c ∈ Icc a b := by
  split_ands
  bound
  bound
have hca : a ≠ c := by
  intro c_eq_a
  rewrite [← c_eq_a] at fc
  linarith [hfa, fc]
have hcb : b ≠ c := by
  intro c_eq_b
  rewrite [← c_eq_b] at fc
  linarith [fc, hfb]
have hcc : c ∈ Ioo a b := by
  split_ands
  sorry
  sorry
use c, hcc, fc
```

Applications and Consequences

The Intermediate Value Theorem has profound applications throughout mathematics:

- **Root-Finding Algorithms**: The bisection method for finding roots is based directly on this theorem
- **Fixed Point Theorems**: Many fixed point theorems use IVT as a crucial step
- **Topological Connectedness**: IVT characterizes the connectedness of intervals
- **Existence Theorems**: Many existence proofs in analysis rely on IVT or its generalizations

Philosophical Significance

The Intermediate Value Theorem captures something deep about the nature of continuous functions and the real line. It says that continuous functions cannot have "jumps"—if a function takes two different values, it must take every value in between.

This seemingly obvious property actually depends on the completeness of the real numbers. In the rational numbers $\mathbb{Q}$, there are continuous functions (like $f(x) = x^2 - 2$) that change sign but never equal zero!

The IVT thus reveals the intimate connection between: - The topological property of continuity - The order structure of the reals - The completeness (Dedekind cuts/Cauchy sequences) of $\mathbb{R}$

This beautiful interplay of analysis, topology, and foundations makes the Intermediate Value Theorem a fitting culmination of our journey through real analysis!

L : Type.

N1)

	R1	R2	R3	...
Hotel:	G1	G2	G3	...

G0

Ledger

R1 G1

R2 G2

$\infty + 1 = \infty$,

N2)

R1	R2	R3	
G1	G2	G3	...

$\infty + 10 = \infty$,

$G_n \to R(n+10)$.

N3)

R1	R2	R3
G1	G2	G3

Bus

...	p3	p2	p1

$\infty + \infty$

N4) $10 \times \infty = \infty$

N5) $\infty \times \infty = \infty$.

N6) ABBA.

Not here!:

BAA...

R1 ABBA

R2 BBAAA ...

R3 ABBA A

...

For length 4, ABBA $\to$ 2^4 names,

Bins has 2^∞ people $> \infty$.

$\mathbb{Z}$, 0, -1, 1, -2, 2, -3, 3, -4, 4, ...

$\mathbb{Q}$. $\frac{p}{q}$. $q =$ Rs #

$\exists$ 1-1, onto map : $\mathbb{N} \to \mathbb{Q}$.

$\mathbb{R} \supset [0,1] \supset \{0.10110\ldots, 0.0110\ldots\}$

these infinities are uncountable.

$2^{\infty} \overset{?}{>} \infty$

Back to Newton!

$$\int_0^1 \sqrt{1-x^2}\,dx = \int 1 - \int \frac{x^2}{2} - \int \frac{x^4}{8} - \int \frac{x^6}{16} - \int \frac{5x^8}{128} - \cdots$$

$f_0(x) = 1$

$f_1(x) = 1 - \frac{x^2}{2}$

$f_2(x) = 1 - \frac{x^2}{2} - \frac{x^4}{8}$

$\ldots$

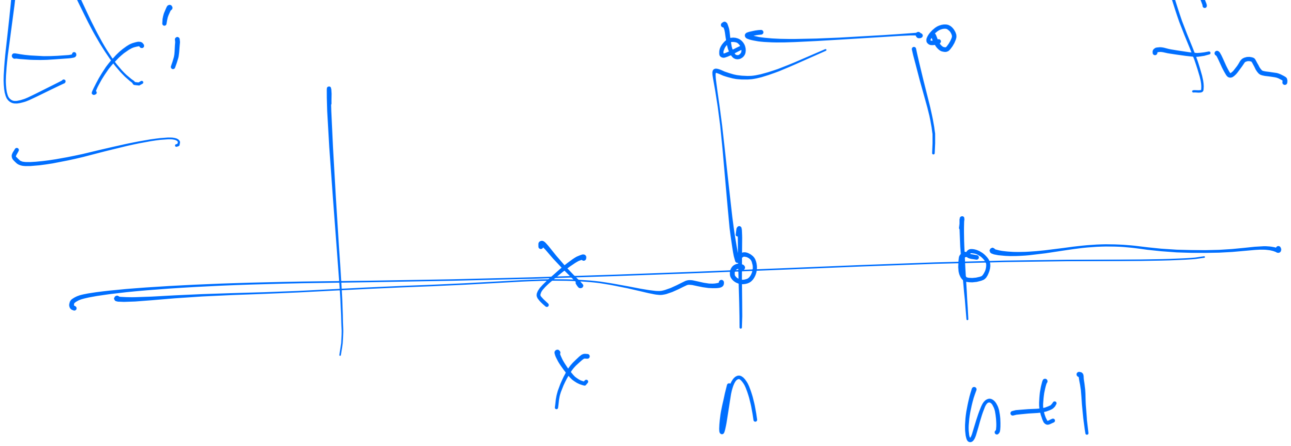
$$\int (f_4) = \int 1 - \int \frac{x^2}{2} - \int \frac{x^4}{8} - \int \frac{5x^8}{16}$$

Q: Given $\{f_n\}$, conv pointwise F.

When is F integrable &

$\int F = \lim \int f_n$?

Ex:



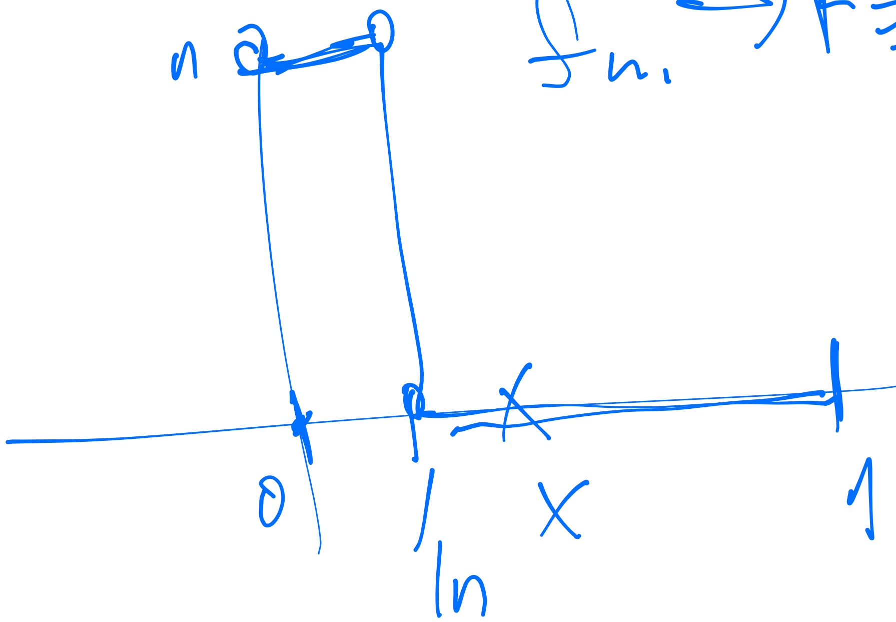
$f_n \to F = 0$, pointwise

$\int f_n = 1$, $\int F = 0$.

Maybe compactness is needed?

Say in $[0,1]$.

Ex 2:

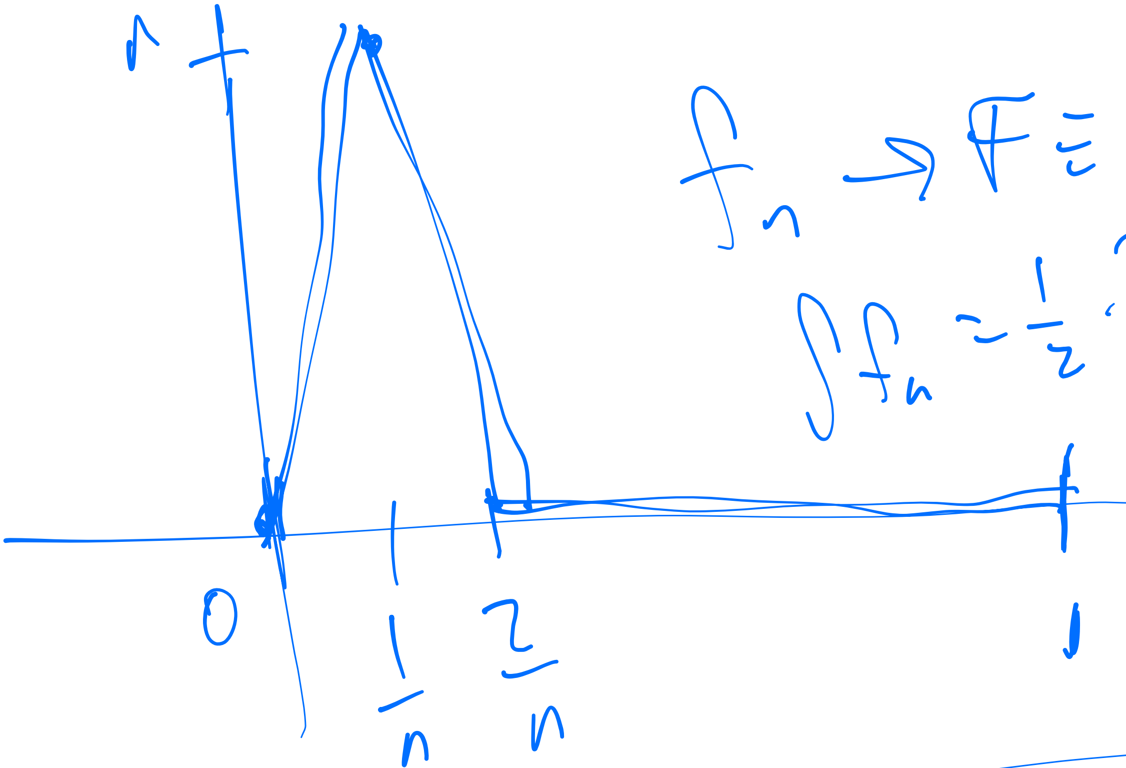


$f_n \xrightarrow{p.w.} F = 0.$

$$\int f_n = n \cdot \frac{1}{n} = 1 \neq \int F = 0.$$

Maybe continuity + compactness is enough?

Ex 3:



$f_n \to F = 0,$

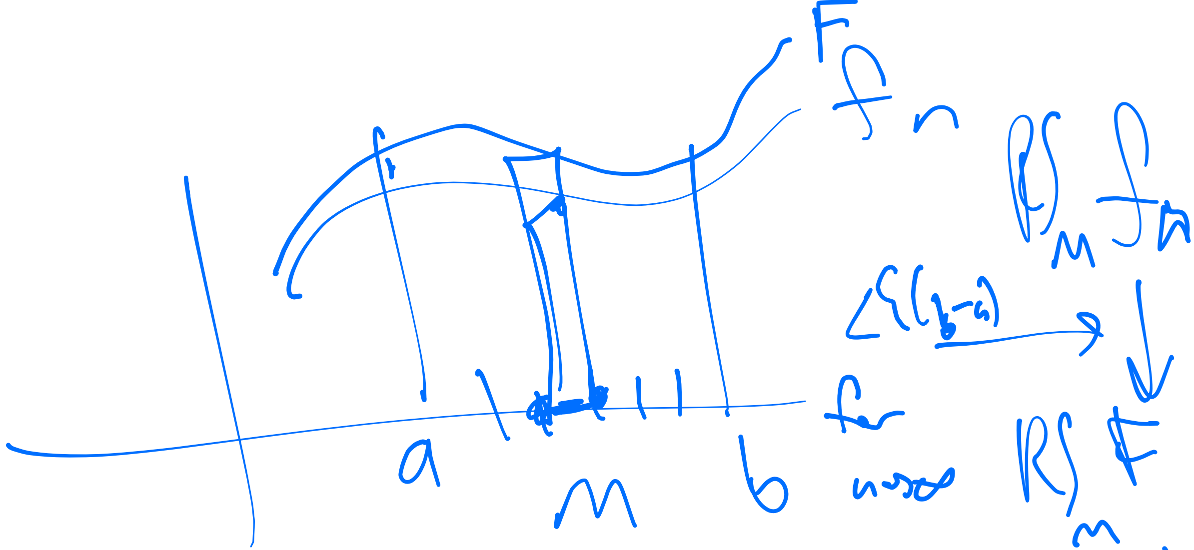
$\int f_n = \frac{1}{2} \cdot \frac{2}{n} n = 1 \neq 0,$

What about uniform Convergence??

$\int f_n \quad , \quad f_n \to F,$

$$RS_M^{[a,b]} f_n - RS_m^{[a,b]} F$$

$$= \frac{b-a}{M} \sum_{i=0}^{M-1} \underbrace{(f_n - F)\left(a + (i+1)\left(\frac{b-a}{m}\right)\right)}_{|\cdot| < \varepsilon}$$



Unif cont: $\forall \varepsilon > 0$, $\exists N$, $\forall n \geq N$, $\forall x$

$$|f_n(x) - F(x)| < \varepsilon.$$

So for any M, as long as $n \to \infty$

Thus $f_n \to F$ uniformly (hfF).

$$\int_a^b f_n = l_n.$$

Goal: $\exists L$, $l_n \to L$ & $\int F = L$.

$$RS_K f_n \Leftrightarrow RS_K F \Leftrightarrow RS_K f_m$$

as $K \to \infty$

$$\ell_n \longleftrightarrow \ell_m$$

Some key: $\forall \varepsilon > 0\ \exists N,\ \forall n \geq N,\ \forall K,$

$$|RS_K f_n - RS_K F| < \varepsilon.$$

$$|RS_K(\underbrace{f_n - F}_{< \frac{\varepsilon}{b-a}})| < \varepsilon.$$

Showed: $\ell_n \to L$.

Part II: $\int F = L$.

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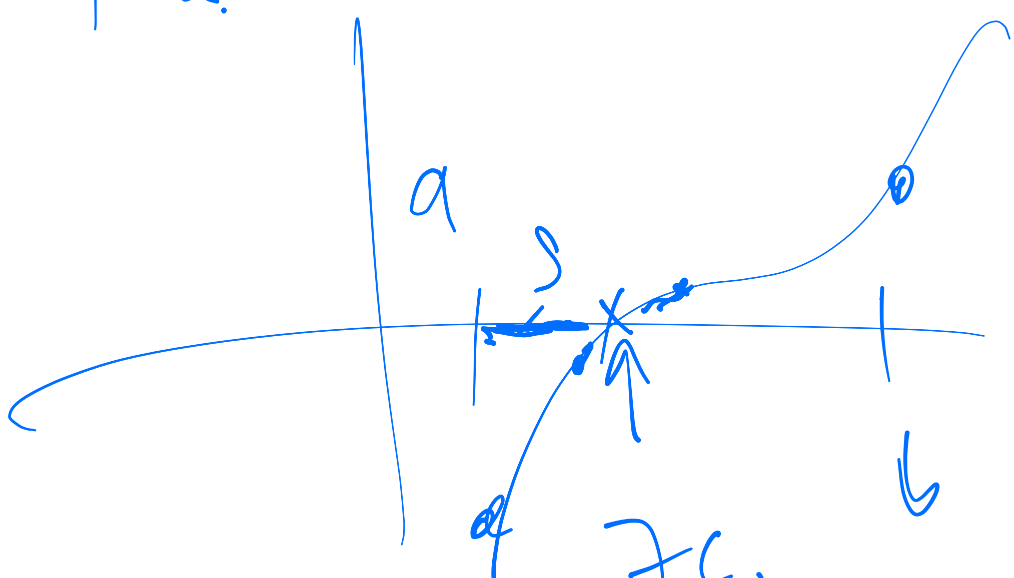
i.e.,

$$\begin{tikzcd}[column sep=huge, row sep=huge]
RS_K F \arrow[r, leftrightarrow, "\checkmark \text{ for } n\to\infty", "N_1"'] \arrow[d, dash, "?"'] & RS_K f_n \arrow[d, dash, "K \text{ lrg},\ <\varepsilon/3,"] \\
L & l_n \arrow[l, "<\varepsilon/3"'] \arrow[l, leftrightarrow, bend left, "n\to\infty", "N_2"']
\end{tikzcd}$$

Fnl Thm:



$\exists c$,

pf: let $S := \{t \in [a,b],\ f(t) < 0\}$,

S nonempty $\ni a$.

S bdd $\leq b$.

$\exists$ LUB for S.

Claim: $c = \text{LUB}$.

Claim $f(c) = 0$.