An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 24: Topology

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SOCRATES: So wait, is "compact" just about being bounded?

SIMPLICIO: Good question! Tell me about the function $f(x) = 1/x$ on the interval $(0, 1)$. It's certainly continuous (as long as we don't include 0). The interval $(0, 1)$ is bounded. If that interval were compact, then f would be uniformly continuous on $(0, 1)$. Is f uniformly continuous on $(0, 1)$?

SOCRATES: Ooh, doesn't look it; the slope gets really steep as x approaches 0. That means that if I want to keep the fluctuation of f, that is, $|f(y) - f(x)|$, small, I need to make $|y - x|$ ever smaller and smaller, for x getting closer to 0. So I can't pick a single δ that works for all x in $(0, 1)$.

SIMPLICIO: Good. Can you see another way to see that $(0, 1)$ is not compact, directly from the definition?

SOCRATES: Umm... I guess I could try to cover $(0, 1)$ with balls. Like, I could cover it with the balls $(1/2, 1) \cup (1/3, 1) \cup (1/4, 1) \cup \ldots$. The Archimedean property (again!!) says that these balls cover all of $(0, 1)$. But if only finitely many of them are used, say up to $(1/N, 1)$, then the point $1/(N + 1)$ is not covered. So there's no finite subcover.

SIMPLICIO: Excellent! So $(0, 1)$ is not compact; that is, being bounded is not enough for compactness. But it's easy to see that bounded *is* necessary. Can you see why?

SOCRATES: Oh, I see it! Let S be our compact set. We can cover S by the balls $(-n, n)$ for all natural numbers n. Since S is compact, there must be a finite subcover. That means that there is some largest N such that $(-N, N)$ covers all of S. Therefore, S is bounded.

SIMPLICIO: Perfect. So boundedness is necessary but, as we just saw with the example of $(0, 1)$, not sufficient for compactness. There's one more ingredient we need.

SOCRATES: What is it?

SIMPLICIO: The set also needs to be *closed*. Here's some more topology-speak for you: A set is *closed* if its complement is *open*.

SOCRATES: Greaaaat, more definitions.

SIMPLICIO: Last one, for now. A set S is *open* if: for every point in S, there is a ball around that point which is entirely contained in S. That is: S is open if:

$$\forall x \in S, \exists \delta > 0, \mathrm{Ball}(x, \delta) \subseteq S$$

Let's check your understanding: is a ball itself open?

SOCRATES: Hmm let's see. I have a point $y \in \mathrm{Ball}(x, r)$. I have to find a $\delta > 0$ such that $\mathrm{Ball}(y, \delta) \subseteq \mathrm{Ball}(x, r)$. The condition that y is in the ball means that $|x - y| < r$, which looks something like this:

```
---|(x-r)----(y)---(x)-------(x+r)|----
```

So if we let $\delta := \min\{y - (x - r), (x + r) - y\}$, then $\mathrm{Ball}(y, \delta)$ will fit inside $\mathrm{Ball}(x, r)$. So yes, a ball is open!

SIMPLICIO: Excellent! So now you have all the ingredients to understand compactness in $\mathbb{R}$: A famous result called the *Heine-Borel Theorem* says that a set $S \subseteq \mathbb{R}$ is compact if and only if:

it is closed and bounded.

Heine proved this in 1872, and Borel generalized it to higher dimensions in 1895. This is a very important theorem in real analysis, because it allows us to easily check whether a set is compact or not. For example, the complement of $[a, b]$ is the union of $(-\infty, a)$ and (b, ∞), both of which are open, so $[a, b]$ is closed. And of course $[a, b]$ is bounded. Therefore, by Heine-Borel, $[a, b]$ is compact. This gives us the important result we wanted:

SOCRATES: Theorem: For any function that is continuous on a closed and bounded interval $[a, b]$, the sequence of Riemann sums converges to a limit, which we call the integral of the function on $[a, b]$.

(Note again that f must be continuous on the *entire* closed interval $[a, b]$. The function $f(x) = 1/x$ is continuous on $(0, 1]$, all but one point; but the Riemann sums do not converge, because $\int_0^1 1/x\, dx = \infty$!)

The proof is very simple: $[a, b]$ is closed and bounded, and hence compact by Heine-Borel. Any function that is continuous on a compact set is uniformly continuous there. And if a function is uniformly continuous on an interval, then the Riemann sums converge to a limit. Just chain everything we've learned together!

SIMPLICIO: Wow, that's really elegant. How is Heine-Borel proved? I guess we already proved half of one direction, if a set is compact then it's bounded. How do we prove that it's also closed?

SOCRATES: You tell me!

SIMPLICIO: Ok, let's try. Suppose S is compact. We want to show that its complement S^c is open. So take any point $x \in S^c$. We need to show that there's a whole ball around x that stays away from S. And the only way to make use of compactness is to cover S with balls. Oh, I think I see it!

For every point $y \in S$, look at the ball centered at y of radius $|y - x|/2$, say. That's a ball that contains y but stays away from x. The collection of all such balls covers S, and hence only finitely many such balls cover S. So we have $V : \mathrm{Finset}(I)$ and for each $i \in V$, we have a point $y_i \in S$ and a ball $\mathrm{Ball}(y_i, |y_i - x|/2)$, and these finitely many balls cover S. Now let $\delta > 0$ be the minimum of all the $|y_i - x|/2$ for $i \in V$. Then the ball $\mathrm{Ball}(x, \delta)$ stays away from all the balls covering S, and hence from S itself. Therefore, S^c is open, and so S is closed.

SOCRATES: Well done! You hardly need me anymore! :) Channel your inner me; what's the next thing I'd say?

SIMPLICIO: You'd tell me to try to prove the other direction. Ok, let's say that S is closed and bounded. We want to show that it's compact. So take any cover of S by balls. We need to find a finite subcover. I... don't see what to do.

SOCRATES: Ok, this direction is a bit harder. Let's build up to it with a few more definitions. (I know, I know.) Given a set S and a real number M, we say that M is an *upper bound* for S if for all $s \in S$, $s \leq M$. Easy, right? We say that L is a *least upper bound* (or *supremum*) for S if L is an upper bound for S, and for any other upper bound M, we have $L \leq M$. In other words, L is the smallest of all upper bounds.

SIMPLICIO: Ok, so far so good... So what?

SOCRATES: Now, here's an important property of the real numbers: *every nonempty set of real numbers that is bounded above has a least upper bound.* This is called the *Least Upper Bound Property.* Let's talk about how you might go about proving it.

SIMPLICIO: Hmm... Let me think. Ok, I start with at least one point $s_0 \in S$, since S is nonempty, and at least one upper bound M_0. I think I see what to do! Let's think about the middle point between s_0 and M_0, that is, $(s_0 + M_0)/2$. Is that an upper bound for S? If not, then there exists some point $s_1 \in S$ such that $s_1 > (s_0 + M_0)/2$. Otherwise, if it *is* an upper bound, then we can set $M_1 := (s_0 + M_0)/2$. In either case, we have a smaller interval $[s_1, M_0]$ or $[s_0, M_1]$. We can keep repeating this process, halving (or more) the interval each time. This gives us a sequence of nested intervals whose lengths go to zero. The bottoms are all points in S and increasing and bounded, hence have a limit L. The tops are all upper bounds and decreasing and bounded, hence have a limit U. Since the lengths of the intervals go to zero, $L = U$. Now, I need to show that L is *the* least upper bound for S. I think I can do that by showing that L is an upper bound, and that any smaller number is not an upper bound. Ok, I'm satisfied.

SOCRATES: Excellent! Now, armed with the Least Upper Bound Property, you can finally prove that any closed and bounded set is compact. First let's prove that a closed interval $[a, b]$ is compact. Take any cover of $[a, b]$ by balls. We want to find a finite subcover. Consider the set T of all points $t \in [a, b]$ such that the interval $[a, t]$ can be covered by finitely many of the balls. Clearly, $a \in T$, so T is nonempty. Also, every point in T is at most b, so T is bounded above by b. By the Least Upper Bound Property, T has a least upper bound L. We want to show that $L = b$. If not, then since the balls cover $[a, b]$, there is some ball covering L. Since the ball has positive radius, it covers some interval around L, say $[L - \delta, L + \delta]$. Since L is the least upper bound of T, there must be some point $t \in T$ with $t > L - \delta$. But then we can cover $[a, t]$ with finitely many balls (since $t \in T$), and also cover $[t, L + \delta]$ with the ball around L. This gives us a finite cover for $[a, L + \delta]$, contradicting the fact that L is an upper bound for T. Therefore, $L = b$, and hence $[a, b]$ can be covered by finitely many balls.

SIMPLICIO: Ok, I'm with you. But what do we do for *any* closed and bounded set?

SOCRATES: Ah, and here's the last step. Any closed subset of a compact set is itself compact! Can you see why?

SIMPLICIO: Hmm, let's see. Let S be a closed subset of a compact set T. Take any cover of S by balls. We want to find a finite subcover. Since S is closed, its complement S^c is open. Therefore, for every point $x \in S^c$, there is a ball around x that stays within S^c, that is, away from S. The collection of all such balls, together with the balls covering S, forms an open cover of the entire set T. Since T is compact, there is a finite subcover of T. This finite subcover must include finitely many balls covering S, since the balls covering S^c do not cover any points in S. Therefore, we have found a finite subcover for S. Hence, S is compact. Nice! Since a bounded set is a subset of some closed interval $[a, b]$, and we've just shown that $[a, b]$ is compact, it follows that any closed and bounded set is compact.

SOCRATES: Well done, Simplicio! You've just proved the Heine-Borel Theorem. And to bring it all the way back to calculus, this means that any continuous function on a closed and bounded interval $[a, b]$ is uniformly continuous there, and hence Riemann integrable. Now let's do all this "for real"...

Level 1: Heine-Borel Theorem: Part 1a

We begin our formal proof of the Heine-Borel theorem by establishing one direction: every compact set is bounded.

The Setup

To prove that a compact set S is bounded, we need to show that there exists some $M > 0$ such that for all $s \in S$, we have $|s| < M$.

The strategy is to use compactness directly: we'll cover S with balls of increasing radius centered at the origin, then use the finite subcover property to extract a bound.

The Key Idea

We cover S with the balls $\mathrm{Ball}(0, n+1)$ for $n \in \mathbb{N}$. By the Archimedean property, these balls cover all of $\mathbb{R}$, so they certainly cover S. Since S is compact, finitely many of these balls suffice to cover S. The largest radius among these finitely many balls gives us our bound.

New Tool: `FinMax`

We need a way to extract the maximum from a finite collection of real numbers:

```
lemma FinMax (ι : Type) (V : Finset ι) (δs : ι → ℝ) :
    ∃ δ, ∀ i ∈ V, δs i ≤ δ
```

This says that any finite collection of real numbers has an upper bound (in fact, a maximum).

The Result

Theorem (`Bdd_of_Compact`): Every compact set is bounded.

Your Challenge

Prove that if S is compact, then $\exists M, \forall s \in S, |s| < M$.

The Formal Proof

```
Statement Bdd_of_Compact (S : Set ℝ) (hcomp : IsCompact
   S) :
    ∃ M, ∀ s ∈ S, |s| < M := by
let ι := ℕ
let xs : ι → ℝ := fun n ↦ 0
let δs : ι → ℝ := fun n ↦ n + 1
have δspos : ∀ n, 0 < δs n := by
  intro n
  change 0 < (n : ℝ) + 1
  linarith
have hcover : S ⊆ ⋃ i, Ball (xs i) (δs i) := by
  intro s hs
  rewrite [mem_Union]
  use ⌈|s|⌉₊
  change s ∈ Ioo ((0 : ℝ) - ((⌈|s|⌉₊ + 1))) (0 + ((⌈|s|⌉
     ₊ + 1)))
  rewrite [mem_Ioo]
  rewrite [show (0 : ℝ) - (⌈|s|⌉₊ + 1) = - (⌈|s|⌉₊ + 1)
     by ring_nf]
  rewrite [show (0 : ℝ) + (⌈|s|⌉₊ + 1) = (⌈|s|⌉₊ + 1) by
      ring_nf]
  rewrite [← abs_lt]
  have f : ∀ x ≥ (0 : ℝ), x ≤ ⌈x⌉₊ := by
    intro x hx
    bound
  specialize f (|s|) (by bound)
  linarith [f]
choose V hV using hcomp ι xs δs δspos hcover
choose M hM using FinMax ι V δs
use M
intro s hs
specialize hV hs
rewrite [mem_Union] at hV
choose i ball_i i_in_V s_in_Ball using hV
rewrite [mem_range] at i_in_V
choose hi hi' using i_in_V
specialize hM i hi
rewrite [← hi'] at s_in_Ball
```

```
change s ∈ Ioo ((0 : ℝ) - (i + 1)) (0 + (i + 1)) at
   s_in_Ball
rewrite [show (0 : ℝ) - (i + 1) = - (i + 1) by ring_nf]
   at s_in_Ball
rewrite [show (0 : ℝ) + (i + 1) = (i + 1) by ring_nf] at
    s_in_Ball
rewrite [mem_Ioo] at s_in_Ball
rewrite [← abs_lt] at s_in_Ball
change i + 1 ≤ M at hM
linarith [s_in_Ball, hM]
```

Understanding the Proof

The proof illustrates the power of compactness: we start with a potentially infinite covering (balls of all possible radii), but compactness guarantees we can reduce to a finite covering. Once we have finitely many balls, we can take their maximum radius to get a global bound.

This is the first half of showing that compactness implies the conjunction "closed and bounded."

Level 2: Heine-Borel Theorem: Part 1b

Now we prove the second half: every compact set is closed. This requires introducing the formal definitions of open and closed sets.

New Definitions

Open Set: S is open if around every point in S, there's an entire ball contained in S:

$$\mathrm{IsOpen}(S) := \forall x \in S, \exists r > 0, \mathrm{Ball}(x, r) \subseteq S$$

Closed Set: S is closed if its complement is open:

$$\mathrm{IsClosed}(S) := \mathrm{IsOpen}(S^c)$$

These definitions capture our intuitive notions: open sets have no "boundary points" that belong to the set, while closed sets contain all their boundary points.

The Strategy

To show that a compact set S is closed, we need to show that S^c is open. This means that for any point $y \in S^c$, we need to find a ball around y that stays entirely within S^c (i.e., doesn't intersect S).

The key insight is to use the separation between y and points in S:

Step 1: For each point $x \in S$, create a ball around x of radius $|y - x|/2$. This ball contains x but doesn't reach y.

Step 2: These balls cover S, so by compactness, finitely many suffice.

Step 3: Take the minimum of the radii $|y - x_i|/2$ from these finitely many balls. This gives a positive δ such that the ball around y of radius δ stays away from all the balls covering S, and hence from S itself.

The Result

Theorem (`IsClosed_of_Compact`): Every compact set is closed.

Your Challenge

Prove that if S is compact, then S^c is open.

The Formal Proof

```
Statement IsClosed_of_Compact (S : Set ℝ) (hcomp :
   IsCompact S) : IsClosed S := by
by_cases Snonempty : S.Nonempty
change IsOpen Sᶜ
intro y hy
change y ∉ S at hy
let ι : Type := S
let xs : ι → ℝ := fun x => x.1
let δs : ι → ℝ := fun x => |y - x.1| / 2
have δspos : ∀ i : ι, δs i > 0 := by
  intro i
  let x : ℝ := i.1
  have hx : x ∈ S := i.2
  have hneq : y ≠ x := by
    intro h
    rw [h] at hy
    contradiction
  have hneq' : y - x ≠ 0 := by bound
  have hdist : |y - x| > 0 := by apply
     abs_pos_of_nonzero hneq'
  bound
have hcover : S ⊆ ⋃ i : ι, Ball (xs i) (δs i) := by
  intro x hx
  rewrite [mem_Union]
  use ⟨x, hx⟩
  change x ∈ Ioo _ _
  rewrite [mem_Ioo]
  specialize δspos ⟨x, hx⟩
  split_ands
  change x - _ < x
  linarith [δspos]
  change x < x + _
  linarith [δspos]
specialize hcomp ι xs δs δspos hcover
choose V hV using hcomp
choose r rpos hr using FinMinPos ι V δs δspos
use r, rpos
intro z hz
```

```
change z ∉ S
intro z_in_S
specialize hV z_in_S
rewrite [mem_Union] at hV
choose i ball_i i_in_V s_in_Ball using hV
change z ∈ Ioo _ _ at hz
rewrite [mem_Ioo] at hz
have hz' : |y - z| < r := by
  rewrite [abs_lt]
  split_ands
  linarith [hz.2]
  linarith [hz.1]
rewrite [mem_range] at i_in_V
choose hi hi' using i_in_V
specialize hr i hi
set ri := δs i
set xi := xs i
let ripos : 0 < ri := by apply δspos i
have hr' : r ≤ ri := by linarith [hr]
have hdist : 2 * ri ≤ |y - xi| := by
  change 2 * (|y - xi| / 2) ≤ |y - xi|
  linarith
have hz'' : |z - xi| < ri := by
  rewrite [← hi'] at s_in_Ball
  change z ∈ Ioo _ _ at s_in_Ball
  rewrite [mem_Ioo] at s_in_Ball
  rewrite [abs_lt]
  split_ands
  linarith [hr, s_in_Ball.1]
  linarith [hr, s_in_Ball.2]
have hz''' : |y - z| ≤ ri := by linarith [hz', hr]
have hiy : |y - xi| ≤ |y - z| + |z - xi| := by
  rewrite [show y - xi = (y - z) + (z - xi) by ring_nf]
  apply abs_add
linarith [hdist, hz'', hz''', hiy, ripos]

intro z hz
push_neg at Snonempty
use 1, (by bound)
intro z hz
```

```
change z ∉ S
rewrite [Snonempty]
intro h
contradiction
```

Understanding the Proof

This proof showcases a classic technique in topology: to separate a point from a set, we first separate it locally from each point in the set, then use compactness to make this separation uniform.

The use of the indexing set $\iota := S$ is elegant - it allows us to parameterize our covering directly by the points of S.

Combined with Level 1, we've now proved that compact $\Rightarrow$ closed and bounded.

Level 3: Least Upper Bound Property

Before proving the converse of Heine-Borel, we need a fundamental property of the real numbers: the Least Upper Bound Property. This property distinguishes $\mathbb{R}$ from $\mathbb{Q}$ and is essential for proving that closed bounded intervals are compact.

New Definitions

Upper Bound: M is an upper bound of S if every element of S is $\leq M$:

$$\text{IsUB}(S, M) := \forall s \in S, s \leq M$$

Least Upper Bound (Supremum): L is a least upper bound of S if L is an upper bound and no smaller number is an upper bound:

$$\text{IsLUB}(S, L) := \text{IsUB}(S, L) \wedge (\forall M, \text{IsUB}(S, M) \rightarrow L \leq M)$$

The Least Upper Bound Property

Theorem: Every nonempty set of real numbers that is bounded above has a least upper bound.

This property can be taken as an axiom defining $\mathbb{R}$, or it can be proved from other characterizations of $\mathbb{R}$ (such as Dedekind cuts or Cauchy completion of $\mathbb{Q}$).

The Proof Strategy

We use a bisection method similar to the one Simplicio outlined in the dialogue:

Step 1: Start with some point $s_0 \in S$ and some upper bound M_0.

Step 2: At each stage, consider the midpoint of our current interval $[s_n, M_n]$. - If the midpoint is an upper bound, make it the new right endpoint - If not, find an element of S above the midpoint and make it the new left endpoint

Step 3: This creates nested intervals whose lengths approach 0.

Step 4: The sequences of left and right endpoints both converge to the same limit L, which is the least upper bound.

The Result

Theorem (`HasLUB_of_BddNonempty`): Every nonempty bounded set has a least upper bound.

Your Challenge

Prove that if S is nonempty and bounded above, then it has a least upper bound.

The Formal Proof

```
Statement HasLUB_of_BddNonempty (S : Set ℝ) (hS : S.
   Nonempty) (M : ℝ) (hM : IsUB S M) : ∃ L, IsLUB S L :=
    by
choose s₀ hs₀ using hS
let ab : ∀ (n : ℕ), {p : ℝ × ℝ //
    (p.1 ∈ S) ∧
    IsUB S p.2 ∧
    p.1 ≤ p.2 ∧
    p.2 - p.1 ≤ (M - s₀) / 2^n} := by
    intro n
    induction' n with n hn
    · use (s₀, M)
      split_ands
      · apply hs₀
      · apply hM
      · bound
      · bound
    · let hp := hn.2
      set p : ℝ × ℝ := hn.1
      let mid : ℝ := (p.1 + p.2) / 2
      by_cases midS : ∃ s ∈ S, mid ≤ s
      · choose s sInS hs using midS
        use (s, p.2)
        split_ands
        · apply sInS
        · apply hp.2.1
        · bound
        · change p.2 - s ≤ (M - s₀) / 2^(n + 1)
```

```
          have hp' := hp.2.2.2
          change (p.1 + p.2) / 2 ≤ s at hs
          field_simp at ⊢ hp' hs
          rewrite [show (2 : ℝ) ^ (n + 1) = 2 * 2 ^ n by
              ring_nf]
          have f : (p.1 + p.2) * 2 ^ n ≤ 2 * s * 2 ^ n
              := by bound
          linarith [hp', hs, hp.2.2.1, f]
      · use (p.1, mid)
        split_ands
        · apply hp.1
        · push_neg at midS
          intro s hs
          linarith [midS s hs]
        · change p.1 ≤ (p.1 + p.2) / 2
          linarith [hp]
        · change (p.1 + p.2) / 2 - p.1 ≤ (M - s₀) / 2^(n
            + 1)
          have hp' := hp.2.2.2
          field_simp at ⊢ hp'
          rewrite [show (2 : ℝ) ^ (n + 1) = 2 * 2 ^ n by
              ring_nf]
          linarith [hp']

let a : ℕ → ℝ := fun n ↦ (ab n).val.1
let b : ℕ → ℝ := fun n ↦ (ab n).val.2

have a_prop : ∀ n, a n ∈ S := by
  intro n
  have h := (ab n).property.1
  apply h

have b_prop : ∀ n, IsUB S (b n) := by
  intro n
  have h := (ab n).property.2.1
  apply h

have aMono : Monotone a := by
  apply Monotone_of_succ
  intro n
```

```
    have h := (ab n).property.2.2.1
    by_cases midS : ∃ s ∈ S, (a n + b n) / 2 ≤ s
    · choose s sInS hs using midS
      have ha' : a (n + 1) = (ab (n + 1)).val.1 := by rfl
      have ha'' : (ab (n + 1)).val.1 = s := by
        sorry
      have f1 : a n ≤ b n := by bound
      linarith [f1, ha', ha'', hs]
    · have ha' : a (n + 1) = a n := by sorry
      linarith [ha']

  have bAnti : Antitone b := by sorry

  have aBnded : ∀ n, a n ≤ b 0 := by
    intro n
    have hb : (b n) ≤ b 0 := by bound
    specialize b_prop n (a n) (a_prop n)
    linarith [b_prop, hb]

  have bBnded : ∀ n, a 0 ≤ b n := by
    intro n
    have ha : a 0 ≤ (a n) := by bound
    apply b_prop n (a 0) (a_prop 0)

  have aCauchy := IsCauchy_of_MonotoneBdd aMono aBnded
  have bCauchy := IsCauchy_of_AntitoneBdd bAnti bBnded

  choose La hLa using SeqConv_of_IsCauchy aCauchy
  choose Lb hLb using SeqConv_of_IsCauchy bCauchy

  have L_le_b : ∀ n, Lb ≤ b n := by sorry

  have L_le_b' : ∀ ε > 0, ∃ N, ∀ n ≥ N, b n < Lb + ε := by
    by_contra h
    push_neg at h
    choose ε εpos hε using h
    choose N hN using hLb ε εpos
    choose n n_N hn using hε N
    specialize hN n n_N
    rewrite [abs_lt] at hN
```

```
  linarith [hN, hn]

have a_le_L : ∀ n, a n ≤ La := by sorry

have a_le_L' : ∀ ε > 0, ∃ N, ∀ n ≥ N, La - ε < a n := by
  by_contra h
  push_neg at h
  choose ε εpos hε using h
  choose N hN using hLa ε εpos
  choose n n_N hn using hε N
  specialize hN n n_N
  rewrite [abs_lt] at hN
  linarith [hN, hn]

have La_eq_Lb : La = Lb := by
  have f1 : SeqLim (fun n ↦ b n - a n) 0 := by sorry
  sorry

use La

split_ands

· intro s hs
  rewrite [La_eq_Lb]
  by_contra h
  push_neg at h
  specialize L_le_b' (s - Lb) (by bound)
  choose N hN using L_le_b'
  specialize hN N (by bound)
  specialize b_prop N s hs
  linarith [hN, b_prop, h]
· intro M hM
  by_contra h
  push_neg at h
  specialize a_le_L' (La - M) (by bound)
  choose N hN using a_le_L'
  specialize hN N (by bound)
  rewrite [show La - (La - M) = M by ring_nf] at hN
  specialize hM (a N) (a_prop N)
  linarith [hM, hN]
```

Understanding the Proof

The least upper bound property is what makes $\mathbb{R}$ "complete" - it has no "gaps" like $\mathbb{Q}$ does. This completeness is precisely what we need to prove that closed bounded intervals are compact.

The bisection proof is constructive and gives us a concrete algorithm for approximating the supremum to arbitrary precision.

Level 4: Heine-Borel Theorem: Part 2a

Now we prove the converse direction of Heine-Borel: every closed bounded interval is compact. This is the hard direction and requires the Least Upper Bound Property.

The Strategy

To show $[a, b]$ is compact, we take an arbitrary covering by balls and show it has a finite subcover. The proof uses a clever "growing interval" approach:

Step 1: Define $S := \{t \in [a, b] : [a, t] \text{ can be covered by finitely many balls}\}$

Step 2: Show that S is nonempty (since $a \in S$) and bounded above (by b)

Step 3: Use the Least Upper Bound Property to get $L = \sup S$

Step 4: Show that $L = b$ by contradiction - if $L < b$, then we can extend our finite cover slightly beyond L, contradicting that L is an upper bound for S

The Key Insight

The crucial idea is that if a point is covered by some ball in our covering, then that entire ball can be covered by a single ball (itself!). So if we can get "close enough" to any point, we can jump all the way to that point and a bit beyond.

The Result

Theorem (`IsCompact_of_ClosedInterval`): Every closed bounded interval $[a, b]$ is compact.

Your Challenge

Prove that $[a, b]$ with $a < b$ is compact.

The Formal Proof

```
Statement IsCompact_of_ClosedInterval {a b : ℝ} (hab : a
    < b) : IsCompact (Icc a b) := by
intro ι xs rs rspos hcover
let S : Set ℝ := {t : ℝ | t ∈ Icc a b ∧ ∃ (J : Finset ι)
   , Icc a t ⊆ ⋃ j ∈ J, Ball (xs j) (rs j)}
have hSnonempty : S.Nonempty := by
  use a
  split_ands
  · bound
  · bound
  · have ha : a ∈ Icc a b := by sorry
    specialize hcover ha
    rewrite [mem_Union] at hcover
    choose j hj using hcover
    use {j}
    intro x hx
    have hxa : x = a := by sorry
    rewrite [hxa]
    use Ball (xs j) (rs j)
    rewrite [mem_range]
    split_ands
    use j
    sorry
    apply hj.1
    apply hj.2
have hSbdd : ∀ s ∈ S, s ≤ b := by
  intro s hs
  exact hs.1.2
choose L hL using HasLUB_of_BddNonempty S hSnonempty b
   hSbdd
have hLb : L = b := by sorry
have hb : b ∈ S := by sorry
simp only [mem_setOf_eq, S] at hb
choose V hV using hb.2
use V, hV
```

Understanding the Proof

This proof is a masterpiece of classical analysis. The key insight is to consider not just individual points, but intervals $[a, t]$ that can be finitely covered. The supremum of all such t must be b, because any point is covered by some ball, and balls have positive radius.

This result shows that the "nice" sets we care about in calculus - closed bounded intervals - are indeed compact.

Level 5: Heine-Borel Theorem: Part 2b

Finally, we complete the Heine-Borel theorem by showing that any closed subset of a compact set is compact. Since bounded sets are subsets of closed intervals, and closed intervals are compact, this will show that closed and bounded sets are compact.

New Tools: Sum Types

We need to work with disjoint unions of types to handle both our original covering and additional balls that avoid the closed set.

Disjoint Union: If α and β are types, then $\alpha \oplus \beta$ represents their disjoint union.

Pattern Matching: We can define functions on $\alpha \oplus \beta$ by cases:

```
let f : α ⊕ β → γ := fun x ↦
  match x with
  | Sum.inl a => ... -- case when x is from α
  | Sum.inr b => ... -- case when x is from β
```

Extracting Components: From a finite set of sum type elements, we can extract just the left components using `Finset.lefts`.

The Strategy

To show that a closed subset S of a compact set T is compact:

Step 1: Start with any covering of S by balls

Step 2: Since S is closed, S^c is open, so we can cover each point in S^c with a ball that stays in S^c

Step 3: The union of these two coverings covers all of T

Step 4: Since T is compact, there's a finite subcover of T

Step 5: Extract just the balls from the original S-covering to get a finite subcover of S

The Result

Theorem (`IsCompact_of_ClosedSubset`): Any closed subset of a compact set is compact.

Your Challenge

Prove that if $S \subseteq T$, T is compact, and S is closed, then S is compact.

The Formal Proof

```
Statement IsCompact_of_ClosedSubset {S T : Set ℝ} (hST :
    S ⊆ T) (hT : IsCompact T) (hS : IsClosed S) :
   IsCompact S := by
intro ι xs rs rspos hcover
change IsOpen Sᶜ at hS
change ∀ x ∈ Sᶜ, ∃ r > 0, Ball x r ⊆ Sᶜ at hS
choose δs δspos hδs using hS
let Sbar : Set ℝ := Sᶜ
let J : Type := Sbar
let U : Type := ι ⊕ J
let xs' : U → ℝ := fun i ↦
  match i with
  | Sum.inl j => xs j
  | Sum.inr x => x
let rs' : U → ℝ := fun i ↦
  match i with
  | Sum.inl j => rs j
  | Sum.inr x => δs x.1 x.2
let rs'pos : ∀ i : U, rs' i > 0 := by
  intro i
  cases i with
  | inl j => exact rspos j
  | inr x => exact δspos x.1 x.2
have hcover' : T ⊆ ⋃ (i : U), Ball (xs' i) (rs' i) := by
  intro t ht
  by_cases htS : t ∈ S
  · specialize hcover htS
    rewrite [mem_Union] at hcover
    choose j hj using hcover
    rewrite [mem_Union]
    use Sum.inl j, hj.1, hj.2
  · change t ∈ Sbar at htS
    rewrite [mem_Union]
```

```
    have hball : t ∈ Ball (xs' (Sum.inr ⟨t, htS⟩)) (rs'
       (Sum.inr ⟨t, htS⟩)) := by
      specialize rs'pos (Sum.inr ⟨t, htS⟩)
      split_ands
      change t - _ < t
      linarith [rs'pos]
      change t < t + _
      linarith [rs'pos]
    use Sum.inr ⟨t, htS⟩, hball.1, hball.2
specialize hT U xs' rs' rs'pos hcover'
choose V hV using hT
let V₁ : Finset ι := V.lefts
use V₁
intro s hs
rewrite [mem_Union]
have hsT : s ∈ T := by bound
specialize hV hsT
rewrite [mem_Union] at hV
choose i ball_i i_in_V s_in_Ball using hV
rewrite [mem_range] at i_in_V
choose hi hi' using i_in_V
rewrite [← hi'] at s_in_Ball
cases i with
| inl j =>
    have hj : j ∈ V₁ := by
      rewrite [mem_lefts]
      apply hi
    use j
    rewrite [mem_Union]
    use hj, s_in_Ball.1, s_in_Ball.2
| inr x =>
    exfalso
    have hxSbar : x.1 ∈ Sbar := x.2
    have hxS : x.1 ∉ S := by
      intro h
      contradiction
    specialize hδs x.1 hxSbar s_in_Ball hs
    apply hδs
```

Understanding the Proof

This proof completes the Heine-Borel theorem. The key insight is that when we have a closed subset of a compact set, we can extend any covering of the subset to a covering of the whole set by adding balls that avoid the subset entirely.

The Complete Heine-Borel Theorem: A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Corollary: Every continuous function on a closed bounded interval is uniformly continuous, and hence Riemann integrable.

This brings us full circle from topology back to calculus, showing how abstract concepts illuminate concrete problems.

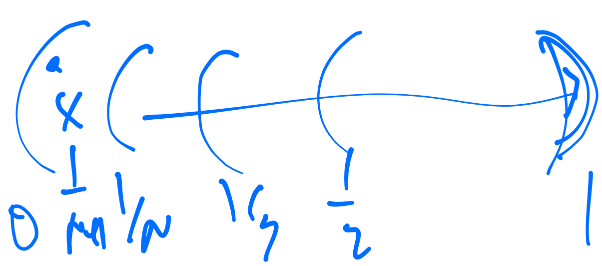
Last time:

Def S is compact, if

"every cover by open balls has finite subcover".

$$\forall\, (\iota : \mathrm{Type})\ (xs : \iota \to \mathbb{R})\ (rs : \iota \to \mathbb{R})\ (rspos : \forall\, i : \iota,\ 0 < rs\ i)\ \Big(hcover : S \subseteq \bigcup_{i : \iota} \mathrm{Ball}\ (xs\ i)\ (rs\ i)\Big),$$

$$\exists\, (V : \mathrm{Finset}\ \iota),\ S \subseteq \bigcup_{i \in V} \mathrm{Ball}\ (xs\ i)\ (rs\ i)$$



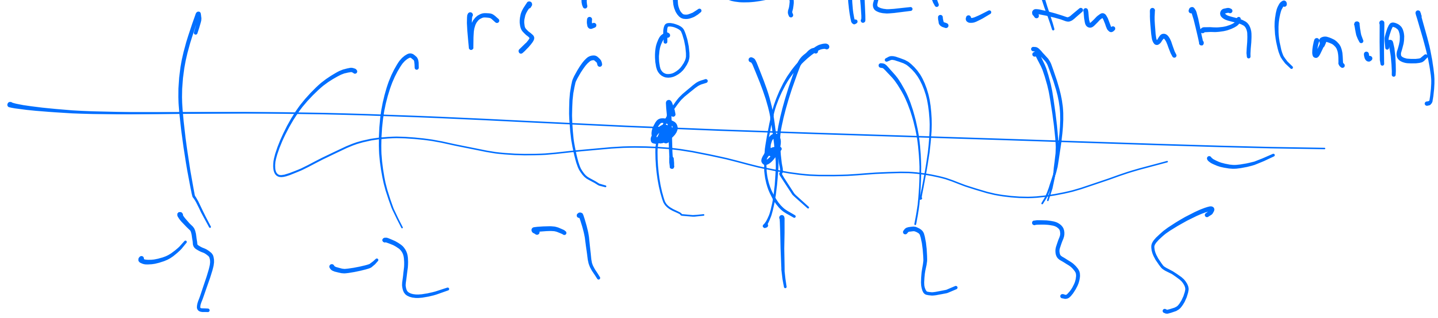
Thm 1: (hS : S compact) $\Rightarrow$ S bdd.

$\exists M, \forall s \in S, |s| \le M.$

Pf: let $\iota := \mathbb{N}$

$x_s : \iota \to \mathbb{R} :=$ fun $n \mapsto (0 : \mathbb{R})$.

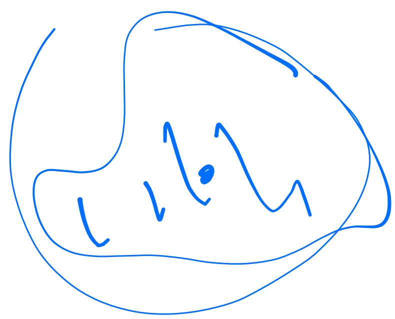
$r_s : \iota \to \mathbb{R} :=$ fun $n \mapsto (n : \mathbb{R})$



Metric space X.

$0 < \text{dist } x\ y = \text{dist } y\ x.$

$\text{dist} : X \to X \to \mathbb{R}.$



take hCover : $\underbrace{S \subseteq \bigcup_n (-n, n)}_{\forall s \in S,\ s \in \bigcup (-n,n)}$.

intro s hs

use $\lceil s \rceil_+$,

(Thm: mem_Union_iff) $s \in \bigcup A_i \iff \exists i, s \in A_i$)

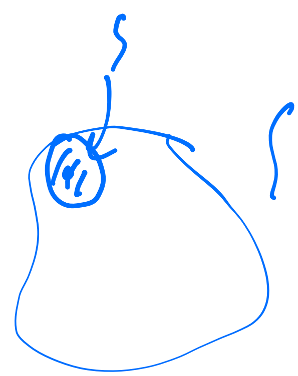
specialize hS $\mathbb{N}$ xs rs rsPos hCover choose V hV using hS.

(Thm: 'FinMax': $\forall$ (ι : Type) (V : Finset ι) (r : $\iota \to \mathbb{R}$) (rs pos) : $\exists$ M, $\forall i \in V$, $r\, i \le M$.)

apply FinMax to get M.
use M.

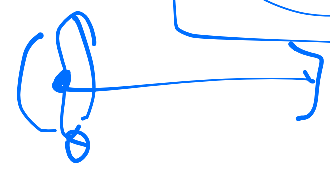
Def: IsClosed (S : Set $\mathbb{R}$) : Prop := IsOpen S^c.

Def: IsOpen - - - - - := $\forall s \in S$, $\exists r > 0$, Ball s r $\subseteq$ S.



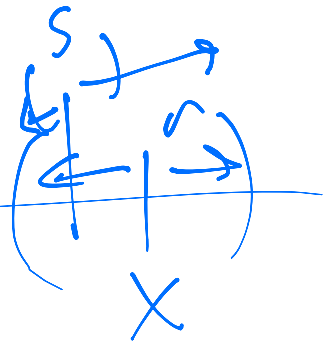
Q: Is [0, 1) open?

No: for s = 0,



Q: Is (a,b) open?

Q: Is $S = \text{Ball}_r(x_0)$ open?



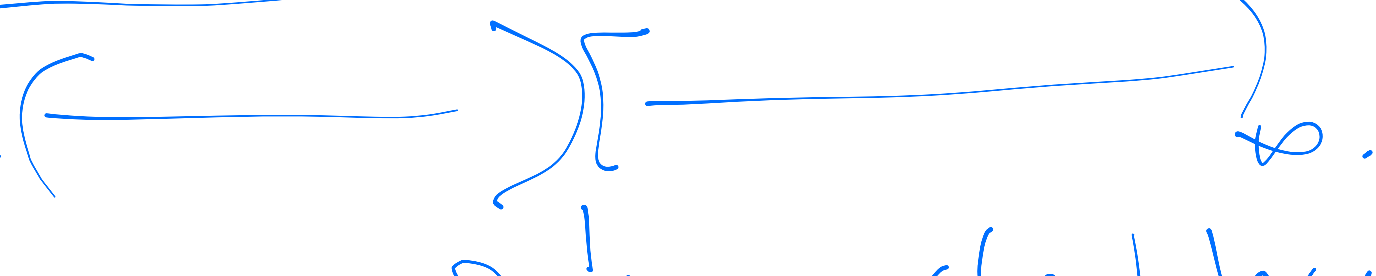
Need to show:

$\forall s \in S, \exists \delta > 0$ s.t. $\text{Ball}_s \delta \subseteq S$.

Thm: Heine-Borel: S is compact

$\iff S$ closed $\wedge$ S bdd.

Ex:



Note: $[1,\infty)$ is closed because $(-\infty, 1)$ is open!

Ex: $[0,1)$ is neither open, nor closed

Ex: If S is closed & open $\Rightarrow S = \emptyset, \mathbb{R}$.

$HB \Rightarrow$ Max Thm: If f is cont on $[a,b]$ then $\int_a^b f(x)\,dx$ exists.

Pf: apply Integrable ⊕ of Unif Cont.

apply Unif Cont of Compact.

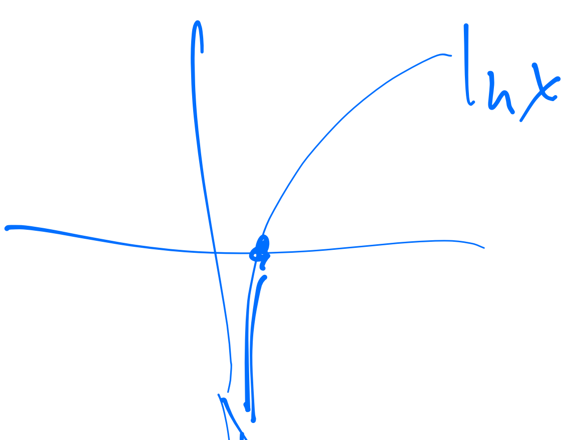
apply HB ($\Rightarrow [a,b]$ is cpt).

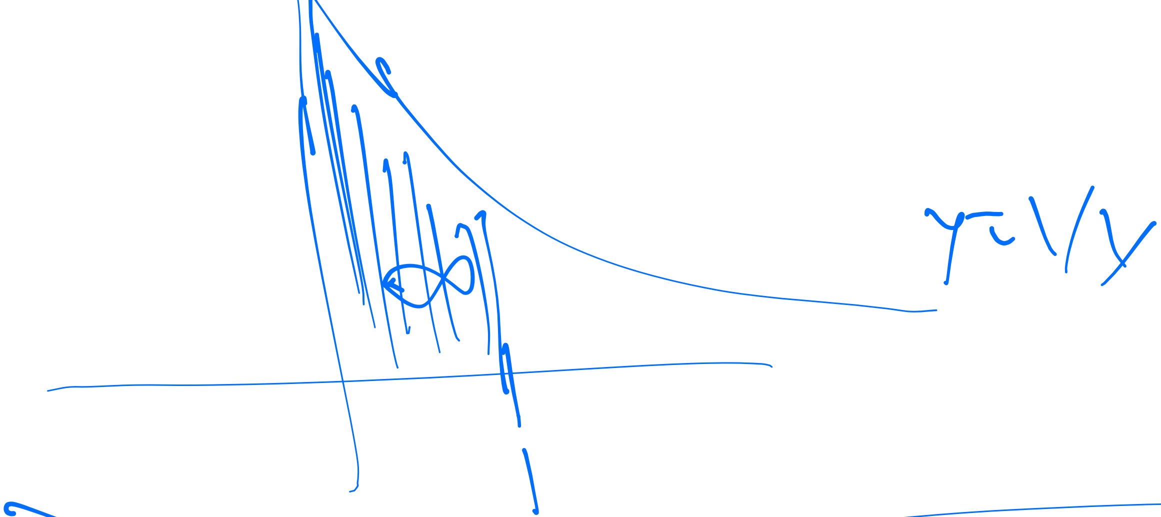
Note: $f(x) = 1/x$ is cont on $(0,1]$,

$$\int_0^1 1/x\,dx = \ln x\Big|_0^1 = \ln 1 - \underbrace{\ln 0^+}_{\to -\infty}$$

$= 0$

$= \infty$





Already proved cpt ⇒ bdd.

Claim: cpt ⇒ closed.

Thm 2: (WS: S cpt)

Goal: Is Closed S.

Pf: change Is Open S^c,

change $\forall y \notin S$, $\exists$ $r>0$ ball $y_r \subseteq S^c$

Idea: Stay away from S!

$L : \text{Type} := S$ (elements of L are $\langle x, hx \rangle$: $x \in S$.)

$XS : L \to \mathbb{R} := \text{fun } s \mapsto s.1$

$rS : L \to \mathbb{R} := \text{fun } i \mapsto |i.1 - y|/2$,

rS_pos.

have hcover : $S \subseteq \bigcup_{i : L} \text{Ball} (XS\ i) (rS\ i)$,

intro s (hs : s ∈ S)

use ⟨s, hs⟩.

---

choose V hV using hS ...

choose δ δpos using finMinPos (V) rS,

use δ

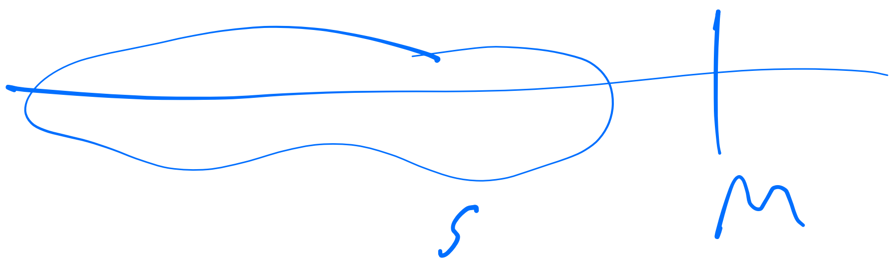
If ∃ z ∈ Ball y δ ∩ S,

$$z \in S \subseteq \bigcup_{i \in V} \text{Ball}\,(x\,;\,i)\,(r\,;\,i).$$

- $|x_i - z| < \delta$.
- $|y - z| < \delta$
- $|x_i - y| = 2r_i \geq 2\delta$

$\lightning$.

HBE: S closed & bdd $\overset{?}{\Rightarrow}$ cpt



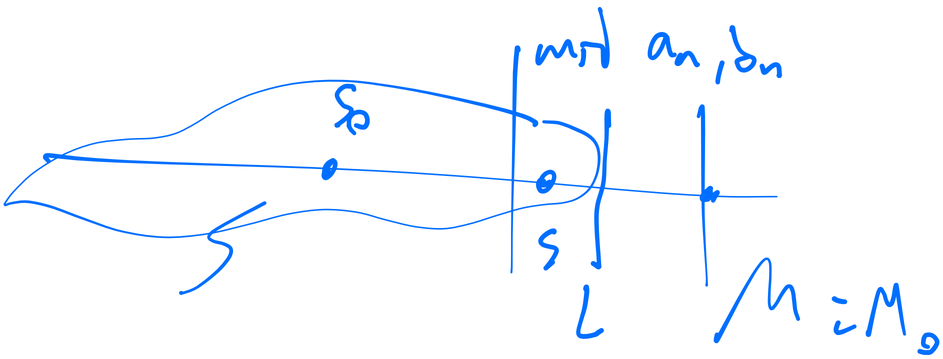
Def: $\text{IsUB}\,(S : \text{Set}\,\mathbb{R})\,(M : \mathbb{R}) : \text{Prop}$

$:= \forall s \in S,\ s \leq M.$

Def, "Supremum": $\text{IsLUB}\,(S)\,(L) :=$

$\text{IsUB}\,L \land \forall M,\ \text{IsUB}\,S\,M \to L \leq M$

Ex: LUB of $(0,1)$ is 1.



$s_n \nearrow \subseteq S$.

$\exists s \in S,\ s > \text{mid}$

$M_n \searrow$ All UB.

If so,

$s_{n+1} = s$, $M_{n+1} = M_n$. $|s_n - M_n| \le \frac{M_0 - s_0}{2^n}$

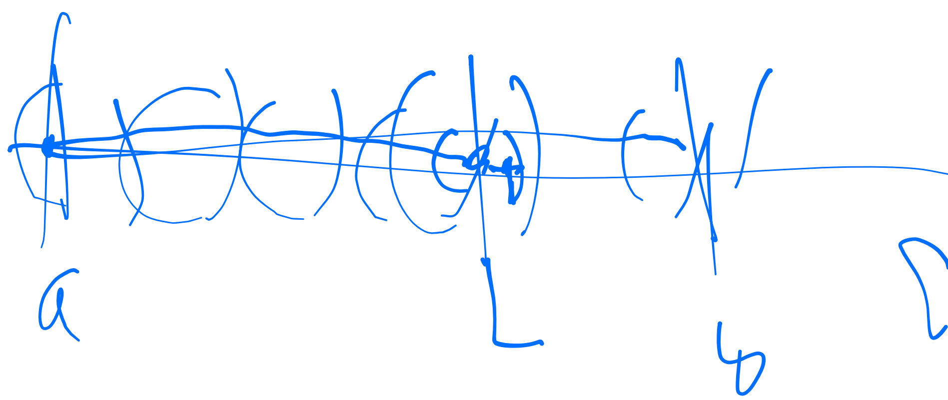
else $s_{n+1} = s_n$, $M_{n+1} = \text{mid}$

By BW, $s_n \to L$, $M_n \searrow K$.

$L = K$. Claim: use L. ✓

Pf $\forall B \in \mathbb{R}$ $[a,b]$ is cpt,

Fix open cover balls,



$[a,a] = a$.

Let $T := \{t \in [a,b] \mid [a,t] \text{ has finite } \overset{sub}{\text{cover}}\}$

nonempty, bdd, $\checkmark$,

$\Rightarrow$ has LUB, L. $\Rightarrow L = b$.

If $L < b$, then L not LUB,

Last step: Any closed subset of $\mathbb{R}^k$

Cpt set is cpt.

Thm 5: S cpt, $T \subseteq S$, T is closed $\Longrightarrow$ T cpt.

Pf: Take open cover of T.

But T^c is open (cover by balls), covers all of $\mathbb{R}$ hence S.

hence can finite cover.

$\Rightarrow$ some finite cover of T.