An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 22: Uniformity

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This text is automatically generated by LLM from "Real Analysis, The Game", Lecture 22

SOCRATES: I just noticed something about that last level.

SIMPLICIO: Ugh. Ok, what was it?

SOCRATES: I don't know, you tell me.

SIMPLICIO: We proved that $x^2 - 1$ was continuous everywhere. So what?

SOCRATES: Right. How did we do it? What δ did we choose, once ε was given?

SIMPLICIO: Are you getting senile, old man? We chose $\delta = \varepsilon/(2|x| + 1)$.

SOCRATES: Anything interesting about that?

SIMPLICIO: What, that it has an x in it? So what? We had no other choice but to choose δ depending on x. We took y near x, $|y - x| < \delta$, and computed that $|f(y) - f(x)|$ was $|y - x| \cdot |y + x|$. The first factor is good, since it's less than δ; in the second factor, since y is near x, then $|y + x|$ has size about $2 \cdot |x|$, and we added one just to be safe.

SOCRATES: Ok, let's put a pin in this and come back to it later. Here's a question: suppose I have a sequence of continuous functions f_n, and suppose f_n converges to some limit function F. That is, for every x, the sequence of real numbers $n \mapsto f_n(x)$ converges to $F(x)$. What can you tell me about F?

SIMPLICIO: Is F continuous? Wait, I've fallen into this trap before. I even remember my counterexample from Lecture 1: Just take $f_n(x) = x^n$ on

$[0, 1]$. Each f_n is continuous, but the limiting function is discontinuous at $x = 1$.

SOCRATES: Exactly! So mere pointwise convergence isn't enough. But let's pretend that it was and see what goes wrong with our proof of continuity.

SIMPLICIO: Ok, so you want me to try (and fail) to prove that F is continuous at some point x. Given $\varepsilon > 0$, we need to find $\delta > 0$ such that for all y with $|y - x| < \delta$, we have $|F(y) - F(x)| < \varepsilon$.

SOCRATES: Right. Go on.

SIMPLICIO: Since f_n converges to F pointwise, for our given x and ε, we can find some big enough N such that for all $n \geq N$, we have $|f_n(x) - F(x)| < \varepsilon/3$.

SOCRATES: Yes. And?

SIMPLICIO: Now, since f_N is continuous at x, we can find some $\delta > 0$ such that for all y with $|y - x| < \delta$, we have $|f_N(y) - f_N(x)| < \varepsilon/3$.

SOCRATES: Good so far. Now, what would you like to do next?

SIMPLICIO: Well, I want to show that $|F(y) - F(x)| < \varepsilon$ for y close to x. I can use the triangle inequality: $|F(y) - F(x)| \leq |F(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - F(x)|$.

SOCRATES: Excellent! And what can you say about each of these three terms?

SIMPLICIO: Well, the middle term is less than $\varepsilon/3$ by our choice of δ. The last term is less than $\varepsilon/3$ by our choice of N. So if I can make the first term less than $\varepsilon/3$, I'm done!

SOCRATES: And can you?

SIMPLICIO: Hmm... I need $|F(y) - f_N(y)| < \varepsilon/3$. Since f_n converges to F at the point y, I can find some M (which might depend on y) such that for $n \geq M$, we have $|f_n(y) - F(y)| < \varepsilon/3$. Uh oh...

SOCRATES: What's the problem?

SIMPLICIO: The problem is that my N was chosen to work at the specific point x, but now I need it to work at this other point y too! And y could be **any** point near x, so I'd need N to work at *all* of these points near x simultaneously.

SOCRATES: Yeah, so what? No matter which y you pick, you can always find some M that works for that y.

SIMPLICIO: But that's exactly the problem! The M I find depends on which y I'm looking at. For one y, I might need $M = 100$. For another y nearby, I might need $M = 1000$. And for yet another y, I might need $M = 10000$.

SOCRATES: So?

SIMPLICIO: So my original N was fixed at the beginning - it only depends on x and ε. But now I need this same fixed N to work for all possible values of y near x. There's no guarantee that my fixed N is bigger than all the different M's I'd need for different y's!

SOCRATES: Ah, I see. So you're saying that even though $f_n(y) \to F(y)$ for each individual y, there might not be a single N that makes the convergence happen "fast enough" simultaneously for all y in a neighborhood?

SIMPLICIO: Exactly! The convergence might be happening at wildly different rates at different points. At some points it might converge quickly, at others very slowly.

SOCRATES: Interesting. So what kind of convergence would you need to make this proof work?

SIMPLICIO: I'd need the convergence to be... uniform over the whole space? Or at least uniform over neighborhoods? So that I can find a single N that works for all points at once, not just point by point.

SOCRATES: Precisely! You've just discovered why we need the concept of **uniform convergence**. Shall we make this precise?

SIMPLICIO: Yes! What exactly do we mean by "uniform convergence"?

SOCRATES: You tell me.

SIMPLICIO: Well, I said that I need a single N that works for all points at once. So instead of saying "for each y, there exists M such that for $n \geq M$, we have $|f_n(y) - F(y)| < \varepsilon/3$", I need to say "there exists N such that for all y and all $n \geq N$, we have $|f_n(y) - F(y)| < \varepsilon/3$".

SOCRATES: Exactly! So uniform convergence means: for every $\varepsilon > 0$, there exists N such that for all $n \geq N$ and for all x in our domain, we have $|f_n(x) - F(x)| < \varepsilon$.

SIMPLICIO: Got it! The key difference is the *order of quantifiers*. In pointwise convergence, we have "for all x, there exists N" - the N can depend on x. In uniform convergence, we have "there exists N such that for all x" - the same N must work for every point.

SOCRATES: Perfect! This is *exactly* what Cauchy got **wrong** in his first attempt at proving that limits of continuous functions were continuous; he was missing uniformity! Ready to work on the proof?

SIMPLICIO: Yes, let's do it!

Level 1: Continuous Composition

Some things with continuous functions are easy. (Some things are not; see the next level!)

The Result

Theorem (`Cont_Comp`): The composition of continuous functions is continuous.

If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both continuous functions, then their composition $f \circ g$ is also continuous.

The Intuition

This result makes intuitive sense: if g is continuous at a point x, then small changes in x produce small changes in $g(x)$. Similarly, if f is continuous at $g(x)$, then small changes in $g(x)$ produce small changes in $f(g(x))$. Chaining these together, small changes in x should produce small changes in $(f \circ g)(x) = f(g(x))$.

The Proof Strategy

Given $\varepsilon > 0$, we want to find $\delta > 0$ such that $|x - c| < \delta$ implies $|(f \circ g)(x) - (f \circ g)(c)| < \varepsilon$.

Since $(f \circ g)(x) = f(g(x))$, we need $|f(g(x)) - f(g(c))| < \varepsilon$.

Step 1: Use the continuity of f at $g(c)$ with tolerance ε to get $\varepsilon_1 > 0$ such that $|y - g(c)| < \varepsilon_1$ implies $|f(y) - f(g(c))| < \varepsilon$.

Step 2: Use the continuity of g at c with tolerance ε_1 to get $\delta > 0$ such that $|x - c| < \delta$ implies $|g(x) - g(c)| < \varepsilon_1$.

Step 3: Now if $|x-c| < \delta$, then $|g(x)-g(c)| < \varepsilon_1$, which means $|f(g(x))- f(g(c))| < \varepsilon$.

Your Challenge

Prove that the composition of continuous functions is continuous:

```
FunCont f → FunCont g → FunCont (f ∘g)
```

The Formal Proof

```
Statement Cont_Comp (f g : ℝ → ℝ) (hf : FunCont f) (hg :
    FunCont g) :
    FunCont (f ∘ g) := by
intro x ε hε
choose ε1 ε1pos hε1 using hf (g x) ε hε
choose δ δpos hδ using hg x ε1 ε1pos
use δ, δpos
intro t ht
specialize hδ t ht
apply hε1 (g t) hδ
```

Understanding the Proof

This proof follows exactly the strategy outlined above. We use the continuity of f at the point $g(x)$ to get an intermediate tolerance ε_1, then use the continuity of g at x with this tolerance to get our final δ. The composition property ensures that the chain of approximations works correctly.

Level 2: Uniform Convergence

As we've discussed several times, pointwise convergence of functions is not enough to preserve continuity. However, there is a stronger notion of convergence, called uniform convergence, which does preserve continuity.

The Definition

Definition (`UnifConv`): Let f_n be a sequence of functions, that is $f : \mathbb{N} \to \mathbb{R} \to \mathbb{R}$, and let F be the hypothetical limit function. We say that f_n converges to F uniformly if:

$$\forall \varepsilon > 0, \exists N, \forall n \geq N, \forall x, |f_n(x) - F(x)| < \varepsilon$$

```
def UnifConv (f : ℕ → ℝ → ℝ) (F : ℝ → ℝ) : Prop :=
  ∀ ε > 0, ∃ N, ∀ n ≥ N, ∀ x, |f n x - F x| < ε
```

Pointwise vs. Uniform Convergence

The key difference between pointwise and uniform convergence is the order of quantifiers:

Pointwise convergence: $\forall x, \forall \varepsilon > 0, \exists N, \forall n \geq N, |f_n(x) - F(x)| < \varepsilon$

Uniform convergence: $\forall \varepsilon > 0, \exists N, \forall n \geq N, \forall x, |f_n(x) - F(x)| < \varepsilon$

In pointwise convergence, the choice of N can depend on both the point x and the tolerance ε. In uniform convergence, we must find a single N that works for *all* points x simultaneously, given only the tolerance ε.

The Main Theorem

Theorem (`Cont_of_UnifConv`): If a sequence of functions f_n converges uniformly to F, and each f_n is continuous, then F is continuous.

This is the theorem that makes uniform convergence so important: it preserves continuity, whereas pointwise convergence does not.

Proof Strategy: The $\varepsilon/3$ Trick

To prove that F is continuous at a point x, given $\varepsilon > 0$, we want to show $|F(y) - F(x)| < \varepsilon$ for y near x.

We use the triangle inequality to write:

$$|F(y) - F(x)| \leq |F(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - F(x)|$$

Our goal is to make each of these three terms less than $\varepsilon/3$:

Term 1: $|F(y) - f_N(y)| < \varepsilon/3$ - This comes from uniform convergence **Term 2**: $|f_N(y) - f_N(x)| < \varepsilon/3$ - This comes from continuity of f_N **Term 3**: $|f_N(x) - F(x)| < \varepsilon/3$ - This also comes from uniform convergence

The crucial point is that uniform convergence gives us a single N that makes both terms 1 and 3 small *simultaneously* for all points, including our specific x and nearby y.

Your Challenge

Prove that the uniform limit of continuous functions is continuous:

($\forall$ n, `FunCont (f n)`) $\rightarrow$ `UnifConv f F` $\rightarrow$ `FunCont F`

The Formal Proof

```
Statement Cont_of_UnifConv  (f : ℕ → ℝ → ℝ) (hf : ∀ n,
   FunCont (f n))
    (F : ℝ → ℝ) (hfF : UnifConv f F) : FunCont F:= by
intro x ε hε
choose N hN using hfF (ε / 3) (by bound)
choose δ hδ₁ hδ₂ using hf N x (ε / 3) (by bound)
use δ, hδ₁
intro y hy
have h1 : |F y - F x| ≤ |f N y - F y| + |f N y - f N x|
   + |f N x - F x| := by
    rewrite [show F y - F x = (F y - f N y) + ((f N y -
       f N x) + (f N x - F x)) by ring_nf]
    have f1 : |(F y - f N y) + ((f N y - f N x) + (f N x
        - F x))| ≤
        |(F y - f N y)| + |((f N y - f N x) + (f N x - F
            x))| := by apply abs_add
    have f2 : |((f N y - f N x) + (f N x - F x))| ≤ |f N
        y - f N x| + |f N x - F x| :=
        by apply abs_add
    have f3 : |F y - f N y| = |f N y - F y| := by apply
       abs_sub_comm
```

```
    linarith [f1, f2, f3]
have h2 : |f N y - F y| < ε / 3 := by apply hN N (by
   bound) y
have h3 : |f N x - F x| < ε / 3 := by apply hN N (by
   bound) x
have h4 : |f N y - f N x| < ε / 3 := by apply hδ₂ y hy
linarith [h1, h2, h3, h4]
```

Understanding the Proof

The proof follows our $\varepsilon/3$ strategy exactly, but the triangle inequality step (h1) deserves special attention:

Step 1: We use uniform convergence to choose N such that f_N is within $\varepsilon/3$ of F at all points.

Step 2: We use the continuity of f_N to choose δ such that $f_N(y)$ is within $\varepsilon/3$ of $f_N(x)$ when y is within δ of x.

Step 3: The key insight is the algebraic rewrite:

$$F(y) - F(x) = [F(y) - f_N(y)] + [f_N(y) - f_N(x)] + [f_N(x) - F(x)]$$

Step 4: We apply the triangle inequality twice:

$$\begin{aligned} |F(y) - F(x)| &= |[F(y) - f_N(y)] + [f_N(y) - f_N(x) + f_N(x) - F(x)]| && (1) \\ &\leq |F(y) - f_N(y)| + |f_N(y) - f_N(x) + f_N(x) - F(x)| && (2) \\ &\leq |F(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - F(x)| && (3) \end{aligned}$$

Step 5: We also use the symmetry $|F(y) - f_N(y)| = |f_N(y) - F(y)|$ to match our uniform convergence bounds.

Step 6: Finally, each term is bounded by $\varepsilon/3$, giving us $|F(y) - F(x)| < \varepsilon$.

Level 3: Integration

Now we can move on to integration. Let's warm up with definitions that you already know from calculus, and a simple example.

New Definitions

Riemann Sum with *right* endpoints:

$$\text{RiemannSum}(f, a, b, N) = \frac{b-a}{N} \sum_{i=0}^{N-1} f\left(a + \frac{(i+1)(b-a)}{N}\right)$$

```
noncomputable def RiemannSum (f : ℝ → ℝ) (a b : ℝ) (N :
   ℕ) : ℝ :=
  (b - a) / N * ∑ i ∈ range N, f (a + (i + 1) * (b - a)
     / N)
```

HasIntegral: A function f has integral I from a to b if the sequence of Riemann sums converges to I:

```
def HasIntegral (f : ℝ → ℝ) (a b : ℝ) (I : ℝ) : Prop :=
  SeqLim (fun N ↦ RiemannSum f a b N) I
```

IntegrableOn: A function f is integrable on $[a, b]$ if there exists some integral value:

```
def IntegrableOn (f : ℝ → ℝ) (a b : ℝ) : Prop :=
  ∃ I, SeqLim (fun N ↦ RiemannSum f a b N) I
```

Helpful Theorems for Summation

To compute Riemann sums, we'll need several theorems about finite sums:

- `sum_add_distrib`: $\sum_{i \in s}(f(i) + g(i)) = \sum_{i \in s} f(i) + \sum_{i \in s} g(i)$
- `sum_const`: $\sum_{i \in s} c = c \cdot |s|$
- `card_range`: $|\{0, 1, \ldots, n-1\}| = n$
- `sum_div`: $\sum_{i \in s}(f(i)/c) = (\sum_{i \in s} f(i))/c$
- `sum_mul`: $\sum_{i \in s}(f(i) \cdot c) = (\sum_{i \in s} f(i)) \cdot c$
- `sum_range_add_one`: $\sum_{i=0}^{n-1}(i+1) = \frac{n(n+1)}{2}$

Computing $\int_a^b x\,dx$

We want to show that the function $f(x) = x$ is integrable on the interval $[a, b]$ where $a < b$, and compute its integral.

From calculus, we expect:

$$\int_a^b x\,dx = \frac{b^2 - a^2}{2}$$

The Riemann Sum Calculation

The Riemann sum for $f(x) = x$ with N subintervals is:

$$\begin{aligned}
\text{RiemannSum}(x, a, b, N) &= \frac{b-a}{N} \sum_{i=0}^{N-1} \left(a + \frac{(i+1)(b-a)}{N} \right) && (4) \\
&= \frac{b-a}{N} \sum_{i=0}^{N-1} \left(a + \frac{(i+1)(b-a)}{N} \right) && (5) \\
&= \frac{b-a}{N} \left[Na + \frac{b-a}{N} \sum_{i=0}^{N-1} (i+1) \right] && (6) \\
&= (b-a)a + \frac{(b-a)^2}{N^2} \sum_{i=1}^{N} i && (7) \\
&= (b-a)a + \frac{(b-a)^2}{N^2} \cdot \frac{N(N+1)}{2} && (8) \\
&= (b-a)a + \frac{(b-a)^2(N+1)}{2N} && (9) \\
&= (b-a)a + \frac{(b-a)^2}{2} + \frac{(b-a)^2}{2N} && (10)
\end{aligned}$$

As $N \to \infty$, this approaches:

$$(b-a)a + \frac{(b-a)^2}{2} = a(b-a) + \frac{(b-a)^2}{2} = ab - a^2 + \frac{b^2 - 2ab + a^2}{2} = \frac{b^2 - a^2}{2}$$

Your Challenge

Prove that $f(x) = x$ is integrable on $[a, b]$ for $a < b$:

```
IntegrableOn (fun x ↦x)a b
```

Hint: Use $(b^2 - a^2)/2$ as your proposed integral value. The key step is showing that the Riemann sum approaches this limit.

The Formal Proof

```
Statement {a b : ℝ} (hab : a < b) :
    IntegrableOn (fun x ↦ x) a b := by
use (b^2-a^2)/2
intro ε hε
have bnd : 0 < 2 * ε / (b - a) ^ 2 := by bound
have bndinv : 0 < 1 / (2 * ε / (b - a) ^ 2) := by bound
choose N hN using ArchProp bnd
use N
intro n hn
have hn' : (N : ℝ) ≤ n := by exact_mod_cast hn
have Npos : (0 : ℝ) < N := by linarith [bndinv, hN]
have npos : (0 : ℝ) < n := by linarith [Npos, hn']
have f1 : (fun N => RiemannSum (fun x => x) a b N) n - (
   b ^ 2 - a ^ 2) / 2  = (b-a)^2 / (2 * n) := by
  change ((b - a) / n * (∑ i ∈ range n, (a + (i + 1) * (
     b - a) / n))) - (b ^ 2 - a ^ 2) / 2  = _
  rewrite [show ∑ i ∈ range n, (a + (i + 1) * (b - a) /
     n) =
    (∑ i ∈ range n, a) +
    ∑ i ∈ range n, ((i + 1) * (b - a) / n) by apply
       sum_add_distrib]
  rewrite [show ∑ i ∈ range n, a = #(range n) • a by
     apply sum_const]
  rewrite [show #(range n) = n by apply card_range]
  rewrite [show ∑ i ∈ range n, ((i + 1) * (b - a) / n) =
      (∑ i ∈ range n, (i + 1) * (b - a)) / n by rewrite
     [← sum_div]; rfl]
  rewrite [show (∑ i ∈ range n, (i + 1) * (b - a)) / n =
      (∑ i ∈ range n, (i + 1 : ℝ)) * (b - a) / n by
     rewrite [← sum_mul]; rfl]
  rewrite [show ∑ i ∈ range n, ((i : ℝ) + 1) = n * (n +
     1) / 2  by apply sum_range_add_one]
  field_simp
  ring_nf
rewrite [f1]
```

```
have f2 : 0 ≤ (b - a) ^ 2 / (2 * n) := by bound
rewrite [abs_of_nonneg f2]
field_simp
field_simp at hN
have f3 : 2 * ε * N ≤ 2 * ε * n := by bound
rewrite [show 2 * ε * n = 2 * n * ε by ring_nf] at f3
linarith [hN, f3]
```

Understanding the Proof

The proof strategy is to show that the difference between the n-th Riemann sum and $(b^2 - a^2)/2$ is exactly $(b-a)^2/(2n)$, which approaches 0 as $n \to \infty$.

Key steps in the computation (f1):

Step 1: We expand the definition of the Riemann sum and separate the sum using `sum_add_distrib`.

Step 2: We evaluate $\sum_{i=0}^{n-1} a = n \cdot a$ using `sum_const` and `card_range`.

Step 3: We factor out constants from the second sum using `sum_div` and `sum_mul`.

Step 4: We apply the crucial identity $\sum_{i=0}^{n-1}(i+1) = \frac{n(n+1)}{2}$ from `sum_range_add_one`
.

Step 5: Through field simplification and ring normalization, we show that:

$$\text{RiemannSum}(x, a, b, n) - \frac{b^2 - a^2}{2} = \frac{(b-a)^2}{2n}$$

Convergence argument: Since we need $\left|\frac{(b-a)^2}{2n}\right| < \varepsilon$, this is equivalent to $n > \frac{(b-a)^2}{2\varepsilon}$. The Archimedean property guarantees we can find such an N, and the proof shows that for all $n \geq N$, the error bound holds.

This completes the proof that $\int_a^b x\,dx = \frac{b^2-a^2}{2}$ using the formal definition of Riemann integration.

Last time, showed

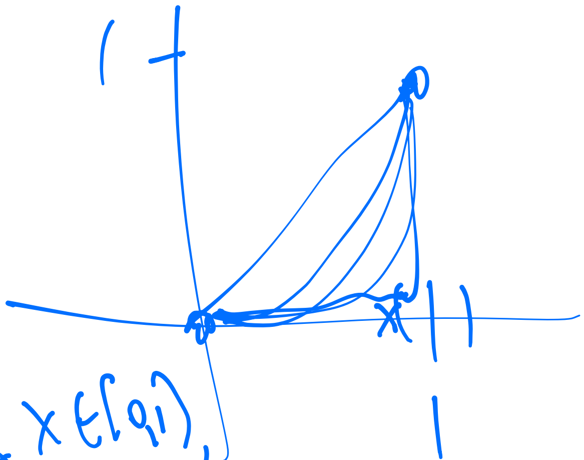
$f(x) = x^2 - 4$ is continuous

What δ did we choose, given $\varepsilon > 0$? $\delta = \dfrac{\varepsilon}{2|x|+1}$

$f_n \to F$ "pointwise" iff

$\forall x$, $\operatorname{Seqlim}(n \mapsto f_n(x))\ F(x)$.

If all f_n's are continuous, then ... ~~X~~.



on $[0,1]$

$f_n(x) = x^n$

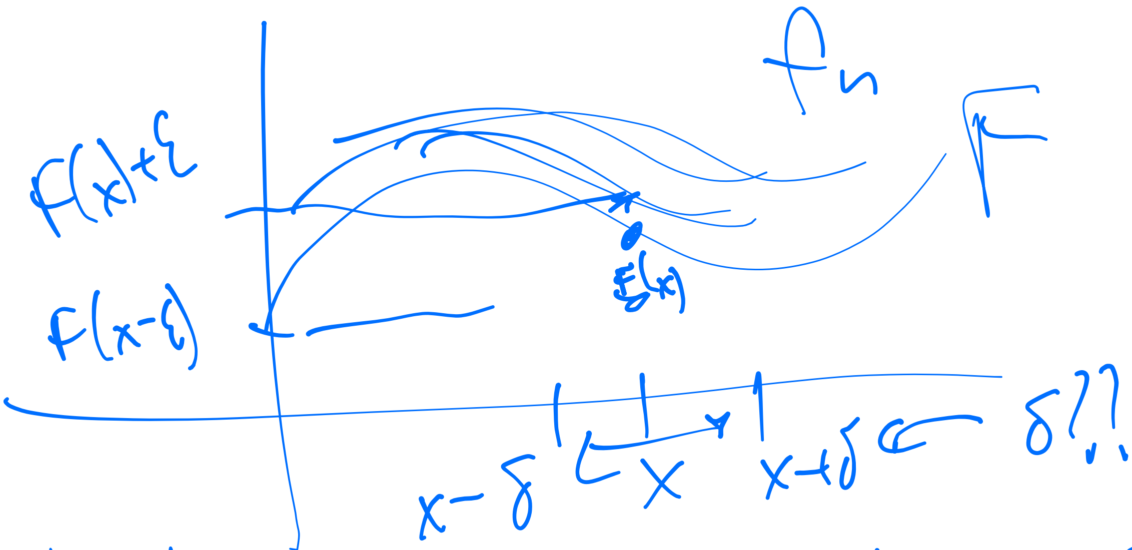
Fix $x \in [0,1]$,

$$\{f_n(x)\} \xrightarrow[n\to\infty]{} \begin{cases} 0 & \text{if } x<1 \\ 1 & \text{if } x=1 \end{cases}$$

"Theorem" ptwise limit of cont functions is cont. (NOT TRUE!)

"pf", Is F cont at x,

let x be given.



Need: $\forall \varepsilon > 0, \exists \delta > 0, \forall |y-x| < \delta, |F(y)-F(x)| < \varepsilon.$

Say we had such a δ, the key question is:

$$|F(y) - F(x)| < \varepsilon ?$$

taking $n \geq N$ (large), $|F(x) - f_n(x)| < \frac{\varepsilon}{3}$,

taking ~~$n \geq N(y)$ large, $|F(y) - f_n(y)| < \frac{\varepsilon}{3}$,~~

But f_n's are cont, so if

$\underbrace{|y-x|<\delta_y}$, $|f_n(y)-f_n(x)|<\frac{\varepsilon}{3}$.

Def. $f_n \to F$ uniformly

If; $\forall \varepsilon>0, \exists N, \forall x, \forall n \geq N,$

$$|f_n(x)-F(x)|<\varepsilon.$$

Def. $f_n \to F$ pointwise

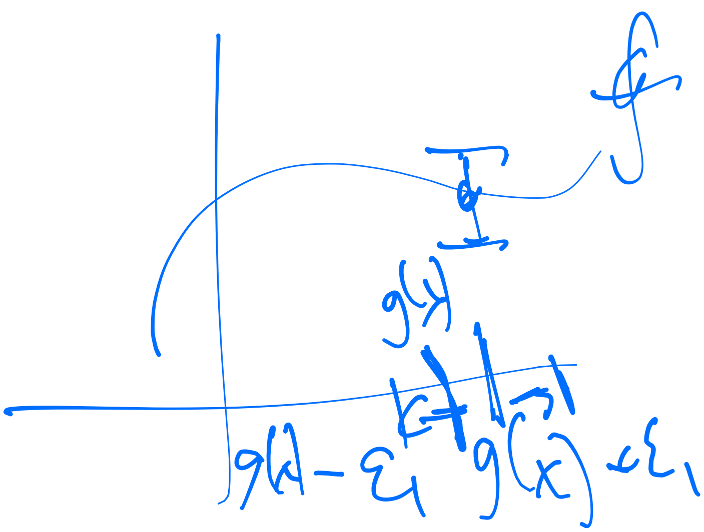
$$\forall \varepsilon>0, \forall x, \exists N, \forall n \geq N,$$

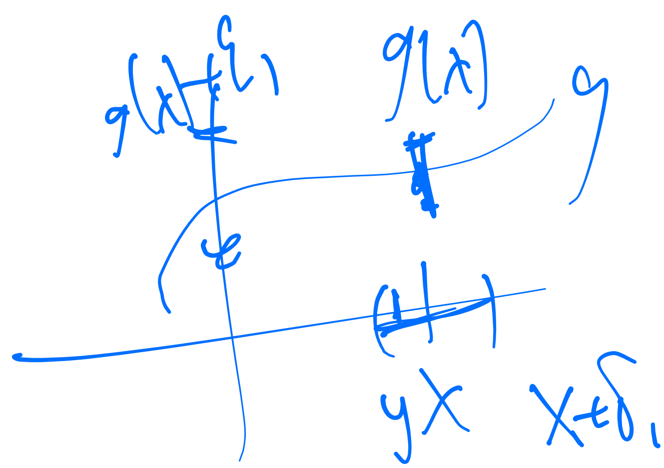
$$|f_n(x)-F(x)|<\varepsilon.$$

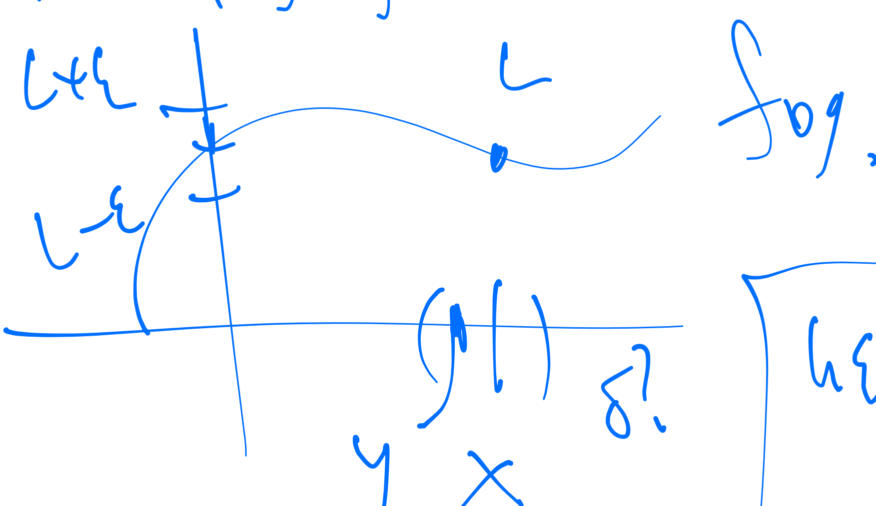
Some things about continuity are not complicated.

E.g.: Thm 'Cont. Comp':

$\forall f, g$: f cont, g cont $\Rightarrow$ cont $(f \circ g)$.







$\boxed{\exists \varepsilon_1 : \forall y,\ |y - g(x)| < \varepsilon_1 \rightarrow |f(y) - f(g(x))| < \varepsilon}$

Pf:

intro x ε $h\varepsilon$

Choose ε_1 $\varepsilon_1 pos$ $h\varepsilon_1$ using hf $(g\ x)$ ε $h\varepsilon$

Choose δ δpos $h\delta$ using hg x ε_1 $\varepsilon_1 pos$

$\boxed{h\delta : \forall y,\ |x-y|<\delta \to |gy - gx| < \varepsilon_1}$

Use δ, δpos.

intro y hy

Specialize $h\delta$ y hy

Specialize $h\varepsilon_1$ $(g\ y)$ $h\delta$

Thm 'Cont_of_UnifConv'

If $f : \mathbb{N} \to \mathbb{R} \to \mathbb{R}$ conv uniformly to $F : \mathbb{R} \to \mathbb{R}$ (hfF) and, $\forall n$, $f\ n$ is cont. (hf)

Then: F is cont.

$$\begin{array}{ccc} f_n(y) & \xleftrightarrow{<\varepsilon/3} & f_n(x) \\ \Big\downarrow {<\varepsilon/3} & & \Big\downarrow {<\varepsilon/3} \end{array}$$

intro x

$$|F(y) - F(x)| < \varepsilon.$$

intro ε hε.

choose N hN using $\varepsilon/3$ ~~hε~~ (by done)

specialize hN N (by done)

choose δ δpos hδ using $\underbrace{\text{hf } N \ x \ (\varepsilon/3) \ (\text{by done})}$

use δ, δpos

intro y hy

have f1 : $|F y - F x| \leq$

$|f N y - F y| +$

$|f N x - F x| +$

$|f N y - f N x|$

have f2: $|f\,N\,y - F\,y| < \varepsilon/3$:= by
  apply hN y

have f3: $|f\,N\,x - f\,x| < \varepsilon/3$:= by
  apply hN x

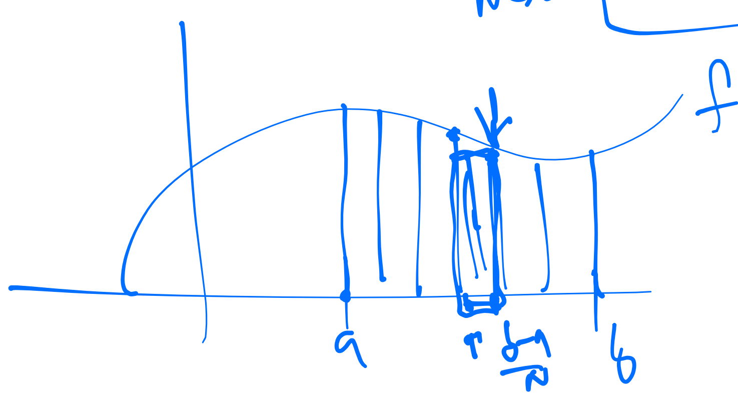
have f4: $|f\,N\,y - f\,N\,x| < \varepsilon/3$:= by
  apply hδ y hy

linarith [f1, .. f4].

Integration!!

$1+2+\dots+n = \dfrac{n(n+1)}{2}$.

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \underbrace{\frac{b-a}{N} \sum_{i=0}^{N-1} f\left(a+(i+1)\cdot\frac{b-a}{N}\right)}_{\text{Riemann Sum}}$$



$$\sum_{i=0}^{n-1} i = \frac{(n-1)(n)}{2}.$$

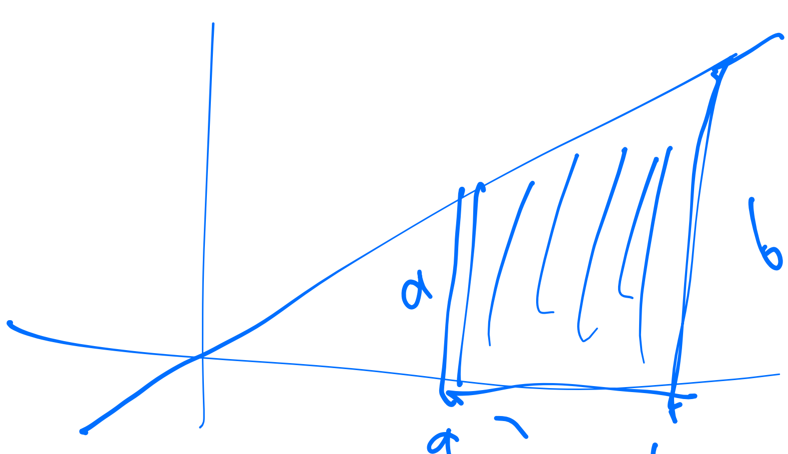
$\int_a^b f\,dx = I \iff$ 'HasIntegral f a b I'

'IntegrableOn f a b' : $\exists I, \int_a^b f\,dx = I$,

<u>Thm</u> (Cauchy $a < b$) IntegrableOn id a b

<u>Sketch</u>: $\int_a^b x\,dx = \underbrace{\frac{x^2}{2}\Big|_a^b = \frac{b^2}{2} - \frac{a^2}{2}}$

$= (b-a)\left(\frac{a+b}{2}\right)$



Use $(b^2-a^2)/2$.
Use ε δ ...

Want: $\left| \underbrace{\frac{b-a}{N} \sum_{i=0}^{N-1} \left(a + \underbrace{(i+1)\frac{b-a}{N}} \right)} - \frac{b^2-a^2}{2} \right| < \varepsilon$

$$= \frac{b-a}{N}(Na) + \frac{(b-a)^2}{N^2} \underbrace{\sum_{i=0}^{N-1} (i+1)}_{= \frac{N^2+N}{2}}$$

$$= \left| \underbrace{a(b-a)} + \underbrace{\frac{(b-a)^2}{2}\left(1 + \frac{1}{N}\right)} - \underbrace{\left(\frac{b^2-a^2}{2}\right)} \right|$$

$$a(b-a) + \frac{(b-a)^2}{2} - \left(\frac{b^2-a^2}{2}\right) = 0$$

Choose N s.t. N using ArchProp

(show $\delta < \frac{2\varepsilon}{(b-a)^2}$ by demo)

Pick n h_n,

Rewrite [Mean ___ $: \frac{(b-a)^2}{2n}$],

New Goal: $\frac{(b-a)^2}{2n} < \varepsilon$.

Know: $n \geq N_1$ $\frac{(b-a)^2}{2 \cdot \varepsilon} < N_1 \leq n$.