An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 21: Functions and Derivatives

Prof. Alex Kontorovich

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Level 1: Sequential Criterion for Limits (Backward Direction)

In this level, we prove the converse direction of the sequential criterion for function limits. This powerful theorem establishes that if sequences test the limit and all tests pass, then the function limit exists!

The Sequential Criterion (Backward Direction)

Theorem (`SequentialCriterion_Backward`): Suppose that for *every* sequence (x_n) with $x_n \to c$ and $x_n \neq c$, we have $f(x_n) \to L$. Then $\lim_{x \to c} f(x) = L$ exists.

This says: if sequences **test** the limit and all tests pass, then the function limit exists!

Why This Is Harder

The forward direction was straightforward: we had δ from the function limit and used it directly.

For the backward direction, we use a **proof by contradiction**:

- Assume $\lim_{x \to c} f(x) = L$ is false
- Then there exists $\varepsilon > 0$ such that for *every* $\delta > 0$, there exists x with $|x - c| < \delta$, $x \neq c$, but $|f(x) - L| \geq \varepsilon$
- We'll construct a **problematic sequence** by choosing such an x for each $\delta = 1/n$
- This sequence converges to c but $f(x_n)$ does *not* converge to L, contradicting our hypothesis!

The Proof Strategy

Given: For all sequences $x_n \to c$ with $x_n \neq c$, we have $f(x_n) \to L$.

Want: To show $\lim_{x \to c} f(x) = L$, i.e., $\forall \varepsilon > 0, \exists \delta > 0, \forall x \neq c, |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

How (by contradiction):

1. Assume not: $\exists \varepsilon > 0$ such that $\forall \delta > 0$, the implication fails
2. For each n, take $\delta = \frac{1}{n+1}$ and get a counterexample x_n with $|x_n - c| < \frac{1}{n+1}$ and $|f(x_n) - L| \geq \varepsilon$
3. Show $x_n \to c$ (since $|x_n - c| < \frac{1}{n+1} \to 0$)
4. By hypothesis, $f(x_n) \to L$, which contradicts $|f(x_n) - L| \geq \varepsilon$

The Formal Proof

```
Statement {f : ℝ → ℝ} {L c : ℝ}
    (h : ∀ x : ℕ → ℝ, (∀ n, x n ≠ c) → SeqLim x c →
       SeqLim (fun n ↦ f (x n)) L) :
    FunLimAt f L c := by
by_contra hf
change ¬ (∀ ε > 0, ∃ δ > 0, ∀ x ≠ c, |x - c| < δ → |f x
   - L| < ε) at hf
push_neg at hf
choose ε hε hδ using hf
```

```
choose g hg_ne_c hg_lt_δ hg using hδ
let x : ℕ → ℝ := fun n ↦ (g (1 / (n + 1)) (by bound))
have hxc : ∀ n, x n ≠ c := by
    intro n;
    apply (hg_ne_c (1 / (n + 1)) (by bound))
have hx_lim : SeqLim x c := by
    intro δ hδ_pos
    choose N hN using ArchProp hδ_pos
    use N
    intro n hn
    have f : |x n - c| < 1 / (n + 1) := by apply hg_lt_δ (1 / (n + 1)) (by bound)
    have f2 : 1 / (n + 1) ≤ δ := by
        have hn' : (N : ℝ) ≤ n := by norm_cast
        have f2' : 0 < 1 / δ := by bound
        have hN' : (0 : ℝ) < N := by linarith [hN, f2']
        have npos : (0 : ℝ) < n := by linarith [hN', hn']
        have hn'' : (1 : ℝ) / n ≤ 1 / N := by bound
        have hn''' : (1 : ℝ) / (n + 1) ≤ 1 / n := by field_simp; bound
        have ff : (1 : ℝ) / N < δ := by field_simp at ⊢ hN; apply hN
        linarith [hn''', hn'', ff]
    linarith [f, f2]
choose N hN using h x hxc hx_lim ε hε
specialize hN N (by bound)
specialize hg (1 / (N + 1)) (by bound)
linarith [hN, hg]
```

Understanding the Proof

The proof uses the axiom of choice (via `choose`) to construct a problematic sequence. The key steps are:

Step 1: Assume by contradiction that the function limit doesn't exist, giving us ε and a function g that produces counterexamples.

Step 2: Construct the sequence $x_n = g(1/(n+1))$ using these counterexamples.

Step 3: Prove $x_n \to c$ using the Archimedean property and the fact that $|x_n - c| < 1/(n+1)$.

Step 4: Apply the hypothesis to get $f(x_n) \to L$, but this contradicts $|f(x_n) - L| \geq \varepsilon$. $\square$

Level 2: Computing a Derivative

We've studied limits of functions extensively. Now we apply this knowledge to one of the most important concepts in calculus: the **derivative**!

The Definition

Definition (`FunDerivAt`): We say that f has derivative L at c if:

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = L$$

This is written `FunDerivAt f L c`.

Reading the definition: The derivative is the limit of the **difference quotient** as $h \to 0$:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

The Geometric Interpretation

The difference quotient $\frac{f(c+h)-f(c)}{h}$ is the **slope of the secant line** through the points $(c, f(c))$ and $(c+h, f(c+h))$.

As $h \to 0$, these secant lines approach the **tangent line** at $x = c$, and the derivative is the slope of this tangent line!

Computing a Derivative

Let's compute the derivative of $f(x) = x^2 - 1$ at $x = 2$.

We need to find the limit:

$$\begin{aligned}
\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h \to 0} \frac{(2+h)^2 - 1 - 3}{h} && (1) \\
&= \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h} && (2) \\
&= \lim_{h \to 0} \frac{4h + h^2}{h} && (3) \\
&= \lim_{h \to 0} \frac{h(4+h)}{h} && (4) \\
&= \lim_{h \to 0} (4+h) && (5) \\
&= 4 && (6)
\end{aligned}$$

So the derivative is 4!

The Formal Proof

```
def FunDerivAt (f : ℝ → ℝ) (L : ℝ) (c : ℝ) : Prop :=
  FunLimAt (fun h ↦ (f (c + h) - f c) / h) L 0

Statement :
    FunDerivAt (fun x ↦ x^2 - 1) 4 2 := by
intro ε hε
use ε, hε
intro h hh0 hh
change |((((2 + h) ^ 2 - 1) - (2 ^ 2 - 1)) / h) - 4| < ε
rewrite [show (2 + h) ^ 2 - 1 - (2 ^ 2 - 1) = 4 * h + h
   ^ 2 by ring_nf]
rewrite [show (4 * h + h ^ 2) / h = 4 + h by field_simp]
rewrite [show 4 + h - 4 = h - 0 by ring_nf]
apply hh
```

Understanding the Proof

Given $\varepsilon > 0$, we need to find $\delta > 0$ such that for $h \neq 0$ with $|h| < \delta$, we have:

$$\left| \frac{(2+h)^2 - 1 - 3}{h} - 4 \right| < \varepsilon$$

Simplify the difference quotient algebraically:

- $(2+h)^2 - 1 - 3 = 4 + 4h + h^2 - 4 = 4h + h^2$
- So $\frac{(2+h)^2-1-3}{h} = \frac{4h+h^2}{h} = 4 + h$ (for $h \neq 0$)
- Thus $\left|\frac{(2+h)^2-1-3}{h} - 4\right| = |4 + h - 4| = |h|$

Therefore, taking $\delta = \varepsilon$ works perfectly! $\square$

Level 3: The Derivative Function

In the previous level, we computed the derivative of $f(x) = x^2 - 1$ at a single point $x = 2$. Now we'll prove something much more powerful: we'll find the derivative at **every** point!

From Point Derivatives to Derivative Functions

So far, `FunDerivAt f L c` tells us that f has derivative L at the specific point c.

But for most functions, we can compute derivatives at *every* point, giving us a **derivative function**.

The New Definition

Definition (`FunDeriv`): We say that g is the derivative of f (everywhere) if:

$$\forall x,\ f \text{ has derivative } g(x) \text{ at } x$$

This is written `FunDeriv f g`.

In other words: for each point x, the derivative of f at x equals $g(x)$.

The Power Rule

For $f(x) = x^2 - 1$, we'll prove that $f'(x) = 2x$ for all x.

This is an instance of the **power rule**: $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$.

Computing the General Derivative

For arbitrary x, we need:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-1-(x^2-1)}{h} \tag{7}$$
$$= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} \tag{8}$$
$$= \lim_{h\to 0}\frac{2xh+h^2}{h} \tag{9}$$
$$= \lim_{h\to 0}\frac{h(2x+h)}{h} \tag{10}$$
$$= \lim_{h\to 0}(2x+h) \tag{11}$$
$$= 2x \tag{12}$$

The Formal Proof

```
def FunDeriv (f : ℝ → ℝ) (g : ℝ → ℝ) : Prop :=
  ∀ x, FunDerivAt f (g x) x

Statement (f g : ℝ → ℝ) (hf : ∀ x, f x = x ^ 2 - 1) (hg
   : ∀ x, g x = 2 * x) :
    FunDeriv f g := by
intro x
intro ε hε
use ε, hε
intro h hh0 hh
change |((f (x + h) - f (x)) / h) - g x| < ε
rewrite [hf, hf, hg]
rewrite [show (x + h) ^ 2 - 1 - (x ^ 2 - 1) = 2 * x * h
   + h ^ 2 by ring_nf]
rewrite [show (2 * x * h + h ^ 2) / h = 2 * x + h by
   field_simp]
rewrite [show 2 * x + h - 2 * x = h - 0 by ring_nf]
apply hh
```

Understanding the Proof

After introducing x, the proof is very similar to Level 2, but with x instead of 2.

Given $\varepsilon > 0$, we use $\delta = \varepsilon$. For $h \neq 0$ with $|h| < \varepsilon$, we simplify:

- $(x + h)^2 - 1 - (x^2 - 1) = 2xh + h^2$
- $\frac{2xh+h^2}{h} = 2x + h$ (for $h \neq 0$)
- $|(2x + h) - 2x| = |h| < \varepsilon$

This establishes the power rule for quadratic functions! $\square$

Level 4: Continuity Everywhere

Just as we moved from derivatives at a point to derivative functions, we can move from continuity at a point to continuity everywhere!

From Point Continuity to Global Continuity

So far, `FunContAt f c` tells us that f is continuous at the specific point c.

But many functions (like polynomials) are continuous at *every* point.

The New Definition

Definition (`FunCont`): We say that f is **continuous** (everywhere) if:

$$\forall x, f \text{ is continuous at } x$$

This is written `FunCont f`.

In other words: f is continuous at every point in its domain.

Why Polynomials Are Continuous

Intuitively, polynomials like $f(x) = x^2 - 1$ are continuous because:

- You can draw them without lifting your pen
- Small changes in x produce small changes in $f(x)$
- There are no jumps, breaks, or asymptotes

We proved earlier that $x^2 - 1$ is continuous at $x = 2$. Now we'll prove it's continuous **everywhere**!

The Strategy

For any point x, we need to show `FunContAt (fun t ↦ t^2 - 1) x`.

This means: given $\varepsilon > 0$, find $\delta > 0$ such that $|t - x| < \delta$ implies $|f(t) - f(x)| < \varepsilon$.

The algebra is similar to our previous work:

$$f(t) - f(x) = (t^2 - 1) - (x^2 - 1) = t^2 - x^2 = (t - x)(t + x)$$

So $|f(t) - f(x)| = |t - x| \cdot |t + x|$.
If we restrict $|t - x| < 1$, then $|t + x| < |2x| + 1$.
Taking $\delta = \min\left(1, \frac{\varepsilon}{|2x|+1}\right)$ will work!

The Formal Proof

```
def FunCont (f : ℝ → ℝ) : Prop :=
  ∀ x, FunContAt f x

Statement :
    FunCont (fun x ↦ x^2 - 1) := by
intro x
intro ε hε
let δ := min 1 (ε / (|2 * x| + 1))
have δ1 : δ ≤ 1 := by bound
have δ2 : δ ≤ (ε / (|2 * x| + 1)) := by bound
have δpos : 0 < δ := by
  have f1 : 0 ≤ |2 * x| := by bound
  have f2 : 0 < |2 * x| + 1 := by bound
  bound
use δ, δpos
intro t ht
change |t ^ 2 - 1 - (x ^ 2 - 1)| < ε
rewrite [show t ^ 2 - 1 - (x ^ 2 - 1) = (t - x) * (t + x
   ) by ring_nf]
rewrite [show |(t - x) * (t + x)| = |t - x| * |t + x| by
    bound]
have ht1 : |t - x| < 1 := by linarith [ht, δ1]
have ht2 : |t - x| < ε / (|2 * x| + 1) := by linarith [
   ht, δ2]
have ht : |t + x| ≤ |2 * x| + 1 := by
  have ht' : |t + x| ≤ |t - x| + |2 * x| := by
    rewrite [show t + x = t - x + 2 * x by ring_nf]
    have f1 : |t - x + 2 * x| ≤ |t - x| + |2 * x| := by
       apply abs_add
    apply f1
  linarith [ht', ht1]
have ht' : |t - x| * |t + x| ≤ |t - x| * (|2 * x| + 1)
   := by bound
```

```
have ht'' : |t - x| * (|2 * x| + 1) < (ε / (|2 * x| + 1)
    ) * (|2 * x| + 1) := by
  field_simp at ⊢ ht2; apply ht2
have ε1 : (ε / (|2 * x| + 1)) * (|2 * x| + 1) = ε := by
    field_simp
linarith [ht', ht'', ε1]
```

Understanding the Proof

After introducing x, the proof carefully bounds $|t + x|$ in terms of x (not a constant).

We use $\delta = \min\left(1, \frac{\varepsilon}{|2x|+1}\right)$. The key insight is that when $|t - x| < 1$, we have:

$$|t + x| = |(t - x) + 2x| \leq |t - x| + |2x| < 1 + |2x|$$

This gives us the bound $|t + x| \leq |2x| + 1$, which allows us to control the product $|t - x| \cdot |t + x|$.

Finally, we get:

$$|f(t) - f(x)| = |t - x| \cdot |t + x| < \frac{\varepsilon}{|2x| + 1} \cdot (|2x| + 1) = \varepsilon$$

This proves that quadratic polynomials are continuous everywhere! □

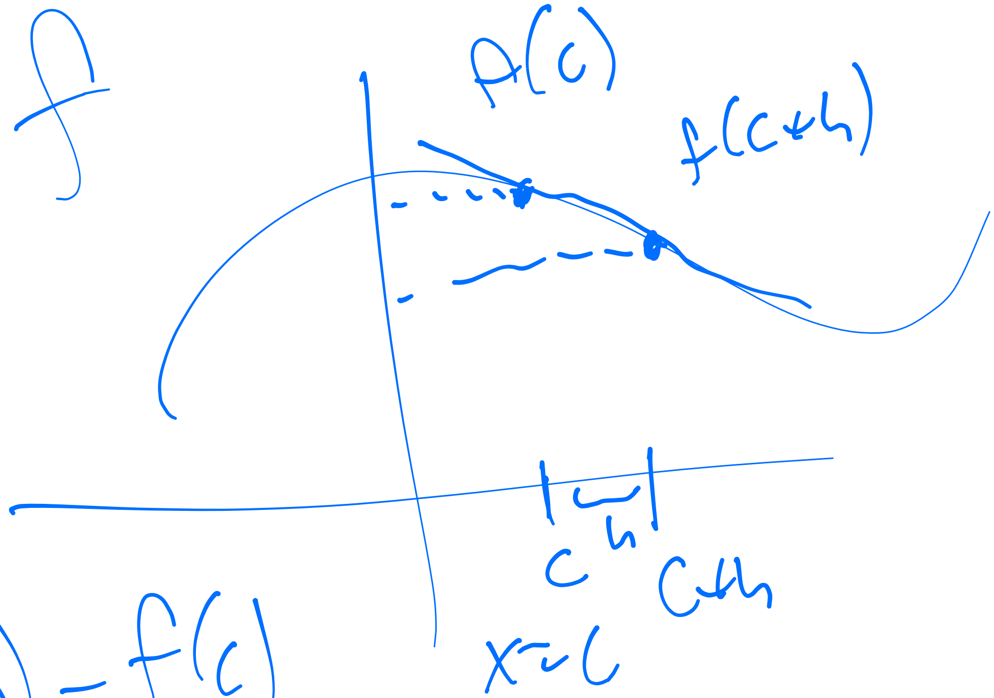
Conclusion

In this lecture, we've established fundamental connections between sequences and functions, computed derivatives using limits, and proven continuity properties. We've seen that:

- The sequential criterion provides a powerful tool for proving function limits
- Derivatives capture the local linear behavior of functions
- Polynomial functions exhibit smooth, continuous behavior everywhere

These results form the foundation for more advanced topics in real analysis and calculus, bridging the discrete world of sequences with the continuous world of functions.

Deriv f



?? wi,

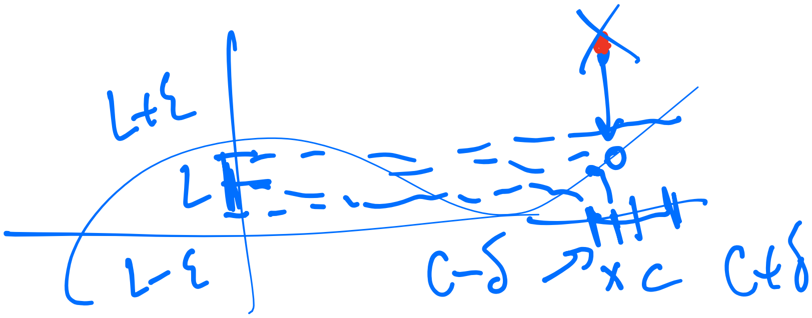
$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} =: f'(c).$$

$h \neq 0,$

FunLimAt $(f : \mathbb{R}\to\mathbb{R})\ (L : \mathbb{R})\ (c : \mathbb{R})$

$: \mathrm{Prop} := \forall \varepsilon > 0, \exists \delta > 0, \forall x \neq c,$

$|x-c| < \delta \to |f\,x - L| < \varepsilon,$



Recall: f is continuous at $x=c$ if: ① $\lim_{x \to c} f(x)$ exists. $(=L)$

② & $f(c) = L$.

Thm: $\exists L$, FunLimAt $\frac{x^2-1}{x-1}$ L 1

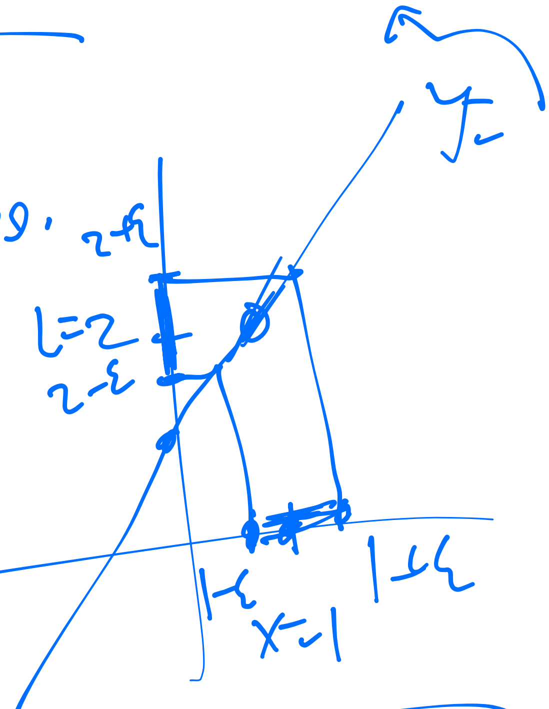
$\varepsilon : \mathbb{R}$

$h\varepsilon : \varepsilon > 0$

Use 2

intro ε $h\varepsilon$.

use ε, $h\varepsilon$.



intro x xNe hx

have f : $x-1 \neq 0$:= by bound

Goal: $\exists \delta > 0$, $\forall x \neq 1$, $|x-1| < \delta \to |f(x)-2| < \varepsilon$.

rewrite [show $\frac{x^2-1}{x-1} = x+1$ by field_simp; ring_nf]

$x : \mathbb{R}$

$xNe : x \neq 1$

$hx : |x-1| < \varepsilon$.

rewrite LHS: $x+1-2$

$= x-1$ by rmy_nf).

apply hx.

Goal: $\left|\frac{x^2-1}{x-1}-2\right| < \varepsilon$.

Goal: $|x+1-2| < \varepsilon$.

Thm: x^2-1 is cont at $x=2$.

Intro ε hε.

let $\delta := \min(\varepsilon/5, 1)$

Use δ, (by dand)

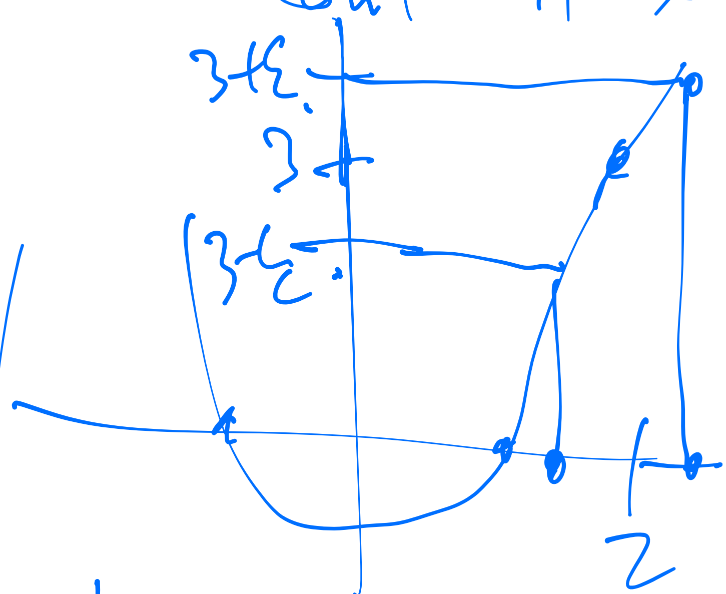
intro x hx

rewrite LHS

$x^2-1-(2^2-1)$

$=(x+2)\cdot(x-2)$

by ring_nf),



$|x^2-1-3| = \underbrace{(x+2)}\underbrace{(x-2)}$ $< 5\delta = \varepsilon$

Note: if $1 \le x \le 3$ then $|x+2| \le 5$

$|f(x)-L| = \underline{(x-c)}\,\underbrace{|g(x)|}_{\le M} \le \delta \cdot M = \varepsilon$.

hx

$\forall x, |x-c| < \delta \to |fx - fc| < \varepsilon.$

rewrite [show]

Goal: $|x^2 - 1 - (2^2 - 1)| < \varepsilon.$

$|(x+2)(x-2)|$

$= |x+2| \cdot |x-2|$ by [done].

$|x+2| \cdot |x-2| < \varepsilon.$

have f1: $|x-2| \le 1$:= by done

$|(x-2)+4|$

$\le |x-2| + 4.$

$|x-2| < 1.$

have f2: $|x-2| \le \varepsilon/5$:= by done.

have f3: $|x+2| \le 5$:= by

rewrite [show $x+2 = x-2+4$ by ring_nf]

have f3': $|x-2+4| \le |x-2| + |4|$:= by apply abs_add

linarith [f3', f1].

have fy: $|x+2| \cdot |x-2| \le 5 \cdot \varepsilon/5$:= by down

linarith [fy].

Thm ContFn AtAdd

$f\ g : \mathbb{R} \to \mathbb{R}$ $\qquad$ hf : FunContAt f c

$c : \mathbb{R}$ $\qquad$ hg : FunContAt g c

Goal: FunContAt $(x \mapsto f\,x + g\,x)$ c,

Let $\varepsilon > 0$ be given.

By hf, we have δ_1 so that

$\forall\ |x-c| < \delta_1$, $|f\,x - f\,c| < \varepsilon/2$.

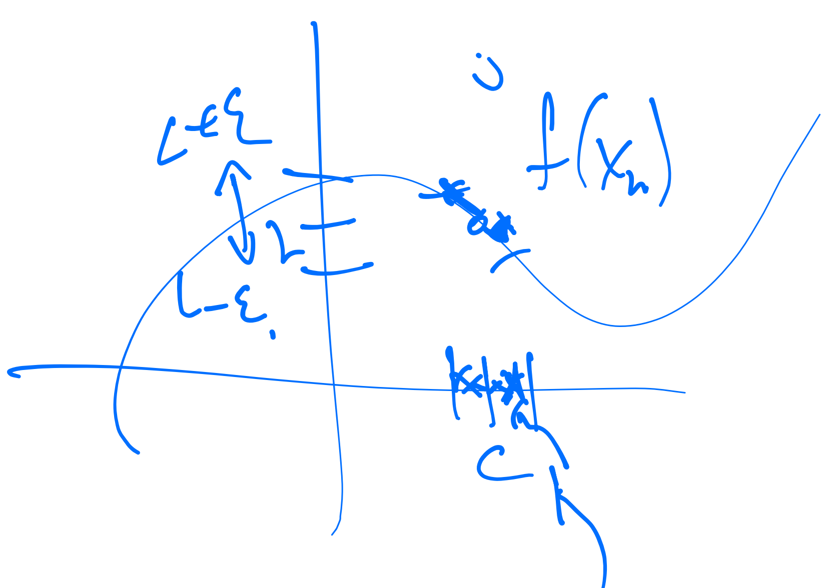
Similarly, choose δ_2 using hg

so that $|x-c| < \delta_2 \Rightarrow |g\,x - g\,c| < \varepsilon/2$, Use $\delta = \min \delta_1, \delta_2$. Then if $|x-c| < \delta$, we need to show that $|fx + gx - (fc + gc)| < \varepsilon$.

We compute: $|fx + gx - (fc + gc)|$
$\le |fx - fc| + |gx - gc| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

<u>Thm:</u> $\underline{f \to L \text{ at } c}$

$\Leftrightarrow$ $\underline{\forall\, x_n \to c}$ $(x_n \neq c)$, $\underline{f(x_n)} \to L$.




c+δ
c
c−δ
x
xₙ
N
n