An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 20: Limits and Continuity of Functions

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SIMPLICIO: FUNCTIONS!!!

SOCRATES: Oh please don't shout, I'm standing right here.

SIMPLICIO: Sorry! I just got a little over excited that we're moving on to functions. Please tell me about them!

SOCRATES: Ah, right. Very good. Let's start at the beginning. What's the first thing you learn in Calculus?

SIMPLICIO: Hmmm. The derivative?

SOCRATES: Ok, we can see about starting there. Tell me, what does it mean to compute the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at some point $x = c$.

SIMPLICIO: Well, it's the slope of the tangent line.

SOCRATES: Yes, of course; I mean, what expression are you trying to evaluate?

SIMPLICIO: Oh! I remember: it's the limit as $h \to 0$ of $(f(x+h)-f(x))/h$.

SOCRATES: Which word there is problematic?

SIMPLICIO: Ah, of course; "limit"! We don't know yet what limits are for functions. And I already know from experience that you can't just stick in $h = 0$, since both the numerator and denominator vanish.

SOCRATES: Right. So we need to figure out what it means for a limit of a *function* to exist. Let's think about this carefully. What do we want to be true when we write $\lim_{x \to c} f(x) = L$?

SIMPLICIO: Well, we want $f(x)$ to get close to L when x gets close to c.

SOCRATES: Exactly! Now, do you remember our Engineer and Machinist from when we discussed sequence limits?

SIMPLICIO: Yes! The Engineer specified a tolerance $\varepsilon > 0$ for how close the output needed to be, and the Machinist replied with how many steps N were needed to guarantee that tolerance.

SOCRATES: Perfect! Now with functions, there's a beautiful twist. The Engineer still specifies a tolerance $\varepsilon > 0$ for the output—that is, we want $|f(x) - L| < \varepsilon$. But what do you think the Machinist's response should be this time?

SIMPLICIO: Hmm. With sequences, the Machinist said "run the process for at least N steps." But with functions, we don't have "steps"... we have values of x.

SOCRATES: Precisely! So instead of saying "wait N steps," how should the Machinist respond?

SIMPLICIO: Oh! So the Machinist needs to give a tolerance on the *input* side, not a number of steps. So he needs to say something like: "make sure your input x is within distance δ of the target point c"?

SOCRATES: Exactly! The conversation goes like this:

- **Engineer:** "I need $f(x)$ to be within ε of L."
- **Machinist:** "No problem! Just make sure your input x is within distance δ of c, and I'll guarantee your output tolerance."

And just like with sequences, we say the limit exists if this conversation can continue for *any* tolerance $\varepsilon > 0$ the Engineer demands—the Machinist can always respond with some appropriate $\delta > 0$.

SIMPLICIO: So the definition would be: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$?

SOCRATES: Beautifully stated! Yes, that's *almost* it. Let's write out what you said formally:

$$\lim_{x \to c} f(x) = L \text{ means: } \forall \varepsilon > 0, \exists \delta > 0, \forall x, |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

This is called an ε-δ definition. There's only one problem with this definition.

SIMPLICIO: Hmm. I really don't see, what's wrong?

SOCRATES: Well, think again back to informal calculus. What does it mean for a function $f : \mathbb{R} \to \mathbb{R}$ to be *continuous* at $x = c$.

SIMPLICIO: Ok, that's when $\lim_{x \to c} f(x)$ exists, and is actually equal to the value of $f(c)$.

SOCRATES: Yes, exactly! Remember when we spoke of derivatives, we don't want to evaluate the limit when h is literally equal to zero, where we get $0/0$. But look again at your definition. Where do you ensure that?

SIMPLICIO: Oh, I see! So we have to update the definition of a limit to make sure that we don't actually allow $x = c$. So does this work?

$$\lim_{x \to c} f(x) = L \text{ means: } \forall \varepsilon > 0, \exists \delta > 0, \forall x \neq c, |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

SOCRATES: That's the ticket! Some people write $0 < |x - c| < \delta$, but I think it'll be easier to just record $x \neq c$ separately. The set of such x is called a *punctured neighborhood* of c—we've removed the center point. This way, the limit only cares about the behavior of f *near* c, not *at* c.

SIMPLICIO: So this means $f(c)$ doesn't even need to be defined for the limit to exist?

SOCRATES: Correct! Remember when you started learning calculus and had to do things like find the limit of $f(x) = \frac{x^2-1}{x-1}$ as x goes to 1?

SIMPLICIO: Yes! This function is undefined at $x = 1$ (actually in Lean, as I've learned, it's perfectly well defined, since $0/0 = 0$ – which means that it's certainly *not* continuous there...). But for $x \neq 1$, we can factor: $f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1$. So $\lim_{x \to 1} f(x) = 2$, even though $f(1)$ doesn't exist!

And this is exactly like the derivative situation. The difference quotient $\frac{f(x+h)-f(x)}{h}$ is undefined at $h = 0$, but we can still take the limit as $h \to 0$.

SOCRATES: Precisely! You've understood the key point. The limit tells us about the *tendency* of a function as we approach a point, not necessarily what happens *at* that point.

SIMPLICIO: So to summarize:

- For limits, we assume $|x - c| < \delta$ **and** $x \neq c$ (punctured neighborhood)
- For continuity, we only need $|x - c| < \delta$; this is equivalent to: the limit exists AND equals $f(c)$.

SOCRATES: You've got it! Let's go.

Level 1: Introduction to Function Limits

Welcome to Lecture 20! We now shift our focus from sequences to **functions**. Just as we studied limits of sequences, we can study limits of functions as the input approaches a particular point.

The Definition

Definition (`FunLimAt`): We say that f has limit L at $x = c$ if:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \neq c, |x - c| < \delta \to |f(x) - L| < \varepsilon$$

This is written `FunLimAt f L c`. (First the function, then the limit, then "at" $x = c$.)

```
def FunLimAt (f : ℝ → ℝ) (L : ℝ) (c : ℝ) : Prop :=
  ∀ ε > 0, ∃ δ > 0, ∀ x ≠ c, |x - c| < δ → |f x - L| < ε
```

Reading the definition: For *every* tolerance ε around the output value L, there exists a distance δ around the input value c such that whenever x is within δ of c (but not equal to c), the function value $f(x)$ is within ε of L.

The Intuition

The key difference from sequence limits is the condition $x \neq c$. We care about what happens *near* c, but not at all about what happens *at* c. The function might not even be defined at c!

This is exactly what happens with the classic example:

$$f(x) = \frac{x^2 - 1}{x - 1}$$

At $x = 1$, the function is "undefined" (because it's $0/0$; in Lean, this is equal to 0). But for $x \neq 1$, we can factor:

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1$$

So as x approaches 1, $f(x)$ approaches some constant L. Your job: figure out L, and prove that it's the limit!

Your Challenge

Prove that there exists a limit L such that:

`FunLimAt (fun x ↦ (x^2 - 1)/(x - 1)) L 1`

In other words, prove that the function $f(x) = \frac{x^2-1}{x-1}$ has *some* limit as x approaches 1.

The Formal Proof

```
Statement :
    ∃ L, FunLimAt (fun x ↦ (x^2 - 1)/(x - 1)) L 1 := by
use 2
intro ε hε
use ε, hε
intro x hxc hx
change |(x ^ 2 - 1) / (x - 1) - 2| < ε
have f1 : x - 1 ≠ 0 := by bound
rewrite [show (x ^ 2 - 1) / (x - 1) = x + 1 by
   field_simp; ring_nf]
rewrite [show x + 1 - 2 = x - 1 by ring_nf]
apply hx
```

Understanding the Proof

Step 1: We use $L = 2$.

Step 2: Given $\varepsilon > 0$, we choose $\delta = \varepsilon$.

Step 3: For any $x \neq 1$ with $|x - 1| < \delta$, we need to show $\left|\frac{x^2-1}{x-1} - 2\right| < \varepsilon$.

Step 4: Since $x \neq 1$, we have $x - 1 \neq 0$, so we can simplify:

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1$$

Step 5: Therefore:

$$\left|\frac{x^2 - 1}{x - 1} - 2\right| = |x + 1 - 2| = |x - 1| < \delta = \varepsilon$$

Thus the limit exists and equals 2. □

Level 2: Continuity at a Point

Excellent work with limits! Now we can define one of the most important concepts in analysis: **continuity**.

The Definition

Definition (`FunContAt`): We say that f is **continuous at** c if:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, |x - c| < \delta \to |f(x) - f(c)| < \varepsilon$$

This is written `FunContAt f c`.

```
def FunContAt (f : ℝ → ℝ) (c : ℝ) : Prop :=
  ∀ ε > 0, ∃ δ > 0, ∀ x, |x - c| < δ → |f x - f c| < ε
```

Continuity vs. Limits

Notice the subtle but crucial differences from `FunLimAt`:

1. **No $x \neq c$ condition:** We care about *all* x near c, including c itself
2. **The limit is $f(c)$:** The function value at c must match the limit as x approaches c; we don't need a separate variable name L for the limit, since L *must* be $f(c)$.

In other words: **A function is continuous at c if its limit at c exists and equals $f(c)$.**

Why This Matters

The function $f(x) = \frac{x^2-1}{x-1}$ from the previous level had a limit at $x = 1$, but it's *not* continuous there (because $f(1) = 0 \neq 2$ in Lean's system).

However, the function $g(x) = x^2 - 1$ is continuous everywhere, including at $x = 2$!

Your Challenge

Prove that the function $f(x) = x^2 - 1$ is continuous at $x = 2$:

`FunContAt (fun x ↦ x² - 1) 2`

Hint: Given $\varepsilon > 0$, you need to find $\delta > 0$ such that $|x - 2| < \delta$ implies $|f(x) - f(2)| < \varepsilon$.

Note that $f(2) = 3$ and $f(x) - f(2) = x^2 - 1 - 3 = x^2 - 4 = (x-2)(x+2)$.

So $|f(x) - f(2)| = |x - 2| \cdot |x + 2|$.

If we restrict x to be within distance 1 of 2 (i.e., $1 < x < 3$), then $|x + 2| < 5$.

Therefore, if we choose $\delta = \min(1, \varepsilon/5)$, we can control $|f(x) - f(2)|$!

The Formal Proof

```
Statement :
    FunContAt (fun x ↦ x^2 - 1) 2 := by
intro ε hε
use min 1 (ε / 5)
split_ands
bound
intro x hx
change |x ^ 2 - 1 - (2 ^ 2 - 1)| < ε
rewrite [show x ^ 2 - 1 - (2 ^ 2 - 1) = (x + 2) * (x -
   2) by ring_nf]
rewrite [show |(x + 2) * (x - 2)| = |(x + 2)| * |(x - 2)
   | by bound]
have f1 : min 1 (ε / 5) ≤ 1 := by bound
have f2 : min 1 (ε / 5) ≤ ε / 5 := by bound
have hx' : |x - 2| < 1 := by bound
have hx'' : |x + 2| < 5 := by
  rewrite [abs_lt] at hx' ⊢
  split_ands
  linarith [hx']
  linarith [hx']
have hx''' : |x - 2| < ε / 5 := by bound
have f3 : |(x + 2)| * |(x - 2)| ≤ 5 * |(x - 2)| := by
   bound
have f4 : 5 * |(x - 2)| < 5 * ε / 5 := by bound
rewrite [show 5 * ε / 5 = ε by bound] at f4
```

```
linarith [f3, f4]
```

Understanding the Proof

Step 1: Given $\varepsilon > 0$, we choose $\delta = \min(1, \varepsilon/5)$. This is positive since both $1 > 0$ and $\varepsilon/5 > 0$.

Step 2: For any x with $|x - 2| < \delta$, we have:

$$|f(x) - f(2)| = |x^2 - 1 - 3| = |(x + 2)(x - 2)| = |x + 2| \cdot |x - 2|$$

Step 3: Since $\delta \leq 1$, we have $|x - 2| < 1$, which implies $1 < x < 3$, so $3 < x + 2 < 5$, giving $|x + 2| < 5$.

Step 4: Since $\delta \leq \varepsilon/5$, we have $|x - 2| < \varepsilon/5$.

Step 5: Therefore:

$$|f(x) - f(2)| = |x + 2| \cdot |x - 2| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon$$

Thus f is continuous at 2. □

Level 3: Sum of Continuous Functions

One of the most powerful aspects of continuity is that it behaves well with respect to algebraic operations. Let's prove our first **continuity theorem**: the sum of continuous functions is continuous!

The Theorem

Theorem (`ContFunAtAdd`): If f and g are both continuous at c, then $f + g$ is continuous at c.

This seems intuitive: if $f(x)$ stays close to $f(c)$ and $g(x)$ stays close to $g(c)$ when x is near c, then their sum should stay close to $f(c) + g(c)$.

The Strategy: The $\varepsilon/2$ Trick

Given $\varepsilon > 0$, we want to make $|(f + g)(x) - (f + g)(c)| < \varepsilon$.

Notice that:

$$\begin{aligned}
|(f + g)(x) - (f + g)(c)| &= |f(x) + g(x) - f(c) - g(c)| \\
&= |[f(x) - f(c)] + [g(x) - g(c)]| \\
&\leq |f(x) - f(c)| + |g(x) - g(c)|
\end{aligned}$$

So if we can make each term less than $\varepsilon/2$, their sum will be less than ε!

Since f is continuous at c, there exists $\delta_1 > 0$ such that $|x - c| < \delta_1$ implies $|f(x) - f(c)| < \varepsilon/2$.

Since g is continuous at c, there exists $\delta_2 > 0$ such that $|x - c| < \delta_2$ implies $|g(x) - g(c)| < \varepsilon/2$.

Taking $\delta = \min(\delta_1, \delta_2)$ ensures both conditions hold simultaneously!

Your Challenge

Prove that if f and g are continuous at c, then their sum is continuous at c:

`FunContAt f c → FunContAt g c → FunContAt (fun x ↦ f x + g x) c`

Hint: After introducing ε and h_ε, use the hypotheses h_f and h_g with $\varepsilon/2$ to choose δ_1 and δ_2. Then use $\min(\delta_1, \delta_2)$. You'll need to show this is positive and that it works. The triangle inequality will be your friend!

The Formal Proof

```
Statement ContFunAtAdd {f g : ℝ → ℝ} {c : ℝ}
    (hf : FunContAt f c) (hg : FunContAt g c) :
    FunContAt (fun x ↦ f x + g x) c := by
intro ε hε
choose δ₁ hδ₁ hf using hf (ε / 2) (by bound)
choose δ₂ hδ₂ hg using hg (ε / 2) (by bound)
use min δ₁ δ₂
split_ands
bound
intro x hx
have hd1 : min δ₁ δ₂ ≤ δ₁ := by bound
have hx1 : |x - c| < δ₁ := by bound
have hd2 : min δ₁ δ₂ ≤ δ₂ := by bound
have hx2 : |x - c| < δ₂ := by bound
specialize hf x hx1
specialize hg x hx2
change |f x + g x - (f c + g c)| < ε
rewrite [show f x + g x - (f c + g c) = (f x - f c) + (g
    x - g c) by ring_nf]
have f1 : |(f x - f c) + (g x - g c)| ≤ |(f x - f c)| +
   |(g x - g c)| := by apply abs_add
linarith [f1, hf, hg]
```

Understanding the Proof

Step 1: Suppose f and g are continuous at c. Given $\varepsilon > 0$, we use the continuity of f at c with $\varepsilon/2$ to obtain $\delta_1 > 0$ such that for all x with $|x - c| < \delta_1$, we have $|f(x) - f(c)| < \varepsilon/2$.

Step 2: Similarly, we use the continuity of g at c with $\varepsilon/2$ to obtain $\delta_2 > 0$ such that for all x with $|x - c| < \delta_2$, we have $|g(x) - g(c)| < \varepsilon/2$.

Step 3: Let $\delta = \min(\delta_1, \delta_2)$. Then $\delta > 0$ since both $\delta_1 > 0$ and $\delta_2 > 0$.

Step 4: For any x with $|x - c| < \delta$, we have:

- $|x - c| < \delta \le \delta_1$, so $|f(x) - f(c)| < \varepsilon/2$
- $|x - c| < \delta \le \delta_2$, so $|g(x) - g(c)| < \varepsilon/2$

Step 5: Therefore, by the triangle inequality:

$$\begin{aligned} |(f+g)(x) - (f+g)(c)| &= |[f(x) - f(c)] + [g(x) - g(c)]| \\ &\leq |f(x) - f(c)| + |g(x) - g(c)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus $f + g$ is continuous at c. $\square$

Level 4: Sequential Criterion for Limits (Forward Direction)

We now have two notions of limits in our arsenal:

1. **Function limits:** `FunLimAt f L c` means $f(x) \to L$ as $x \to c$
2. **Sequence limits:** `SeqLim x L` means $x_n \to L$ as $n \to \infty$

Could these concepts be connected? It's **mathematics**, how could they not!

In this level, we'll prove the first half of the **Sequential Criterion for Limits**.

The Sequential Criterion (Forward Direction)

Theorem: If f has limit L at c, then for *every* sequence (x_n) with $x_n \to c$ and $x_n \neq c$, we have $f(x_n) \to L$.

In other words: function limits can be **tested** using sequences!

Why This Matters

This theorem is incredibly useful because:

- It connects two different limit concepts
- It lets us use sequence intuition to understand function limits
- It may be easier to work with certain sequences than with the ε-δ definition

The Proof Strategy

Given: `FunLimAt f L c` and a sequence (x_n) with $x_n \to c$ and $x_n \neq c$.
Want: To show $f(x_n) \to L$, i.e., for all $\varepsilon > 0$, eventually $|f(x_n) - L| < \varepsilon$.
How:

1. Given $\varepsilon > 0$, use `FunLimAt` to get $\delta > 0$ such that $|x - c| < \delta$ and $x \neq c$ implies $|f(x) - L| < \varepsilon$
2. Use `SeqLimit` to get N such that for all $n \geq N$, we have $|x_n - c| < \delta$
3. For $n \geq N$, we know $x_n \neq c$ and $|x_n - c| < \delta$, so $|f(x_n) - L| < \varepsilon$

Your Challenge

Prove the forward direction of the sequential criterion:

`FunLimAt f L c → (∀ x : ℕ → ℝ, (∀ n, x n ≠ c) → SeqLim x c → SeqLim (fun n ↦ f (x n)) L)`

Hint: After introducing all the hypotheses, introduce ε and h_ε. Use h_f with ε to get δ and its properties. Then use h_x with δ to get N. Use this N to show that the sequence $f(x_n)$ converges to L.

The Formal Proof

```
Statement {f : ℝ → ℝ} {L c : ℝ}
    (hf : FunLimAt f L c) :
    ∀ x : ℕ → ℝ, (∀ n, x n ≠ c) → SeqLim x c → SeqLim (
       fun n ↦ f (x n)) L := by
intro x hxc hx
intro ε hε
choose δ hδ hfδ using hf ε hε
choose N hN using hx δ hδ
use N
intro n hn
specialize hN n hn
specialize hxc n
apply hfδ (x n) hxc hN
```

Understanding the Proof

Step 1: Suppose `FunLimAt f L c` holds. Let $x : \mathbb{N} \to \mathbb{R}$ be a sequence such that $x_n \neq c$ for all n and $x_n \to c$.

Step 2: To show that $f(x_n) \to L$, let $\varepsilon > 0$ be given.

Step 3: Since `FunLimAt f L c`, there exists $\delta > 0$ such that for all x with $x \neq c$ and $|x - c| < \delta$, we have $|f(x) - L| < \varepsilon$.

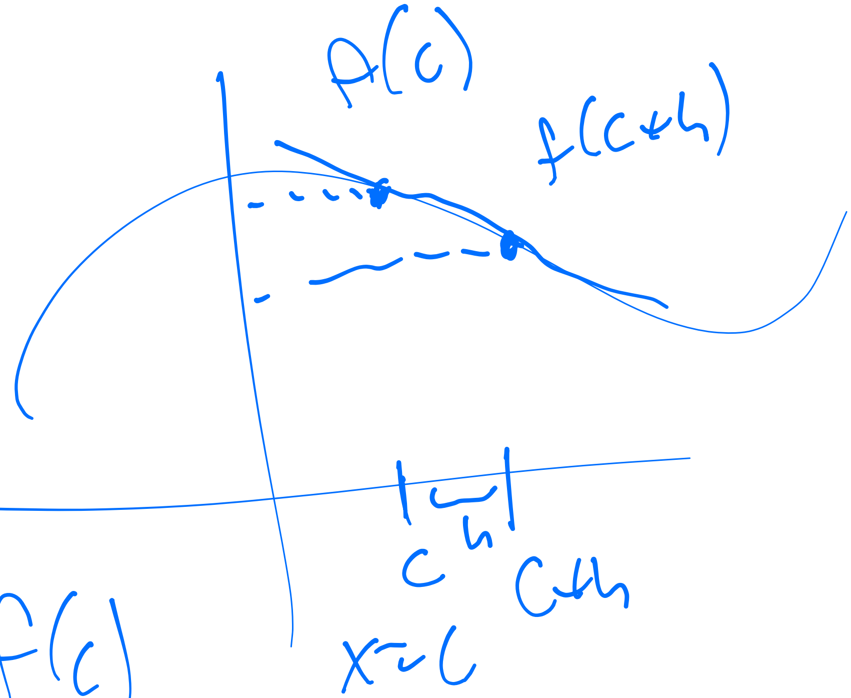
Step 4: Since $x_n \to c$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|x_n - c| < \delta$.

Step 5: For any $n \geq N$, we have:

- $x_n \neq c$ (by hypothesis)
- $|x_n - c| < \delta$ (since $n \geq N$)

Step 6: Therefore, by the definition of `FunLimAt`, we have $|f(x_n) - L| < \varepsilon$. This shows that $f(x_n) \to L$, completing the proof. $\square$

Deriv. f



??

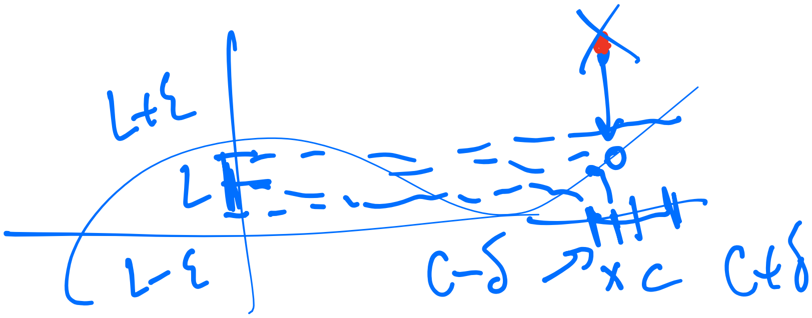
$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} =: f'(c).$$

$h \neq 0$,

$\mathrm{FunLimAt}\ (f : \mathbb{R} \to \mathbb{R})\ (L : \mathbb{R})\ (c : \mathbb{R})$

$: \mathrm{Prop} := \forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \neq c,$

$|x - c| < \delta \to |f\,x - L| < \varepsilon,$



Recall, f is continuous at $x=c$ if: ① $\lim_{x \to c} f(x)$ exists. $(=L)$

② & $f(c) = L$.

Theorem: $\exists L$, FunLimAt $\frac{x^2-1}{x-1}$ L 1

Use 2

Intro ε hε.

use ε, hε.

Intro x xNe hx

have f : x-1 ≠ 0 := by bound

rewrite [show $\frac{x^2-1}{x-1} = x+1$ by field_simp; ring_nf]

ε : ℝ

hε : ε > 0, 2+ε

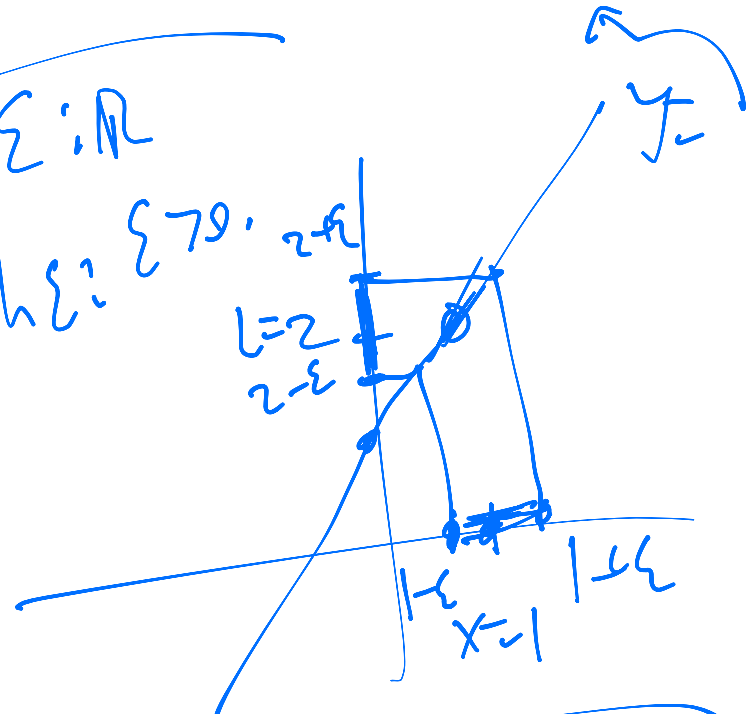
L=2

2-ε

1-ε   1+ε

x=1

y=



∀ x ≠ 1,

Goal: ∃ δ > 0, |x-1| < δ → |f(x) - 2| < ε.

x : ℝ

xNe : x ≠ 1

hx : |x-1| < ε.

rewrite [show x+1-2

= x-1 by ring_nf]?

apply hx.

Goal: $\left|\frac{x^2-1}{x-1} - 2\right| < \varepsilon$.

Goal: $|x+1-2| < \varepsilon$.

Thm: x^2-1 is cont at $x=2$.

intro ε hε.

let δ := min (ε/5) 1

use δ, (by done)

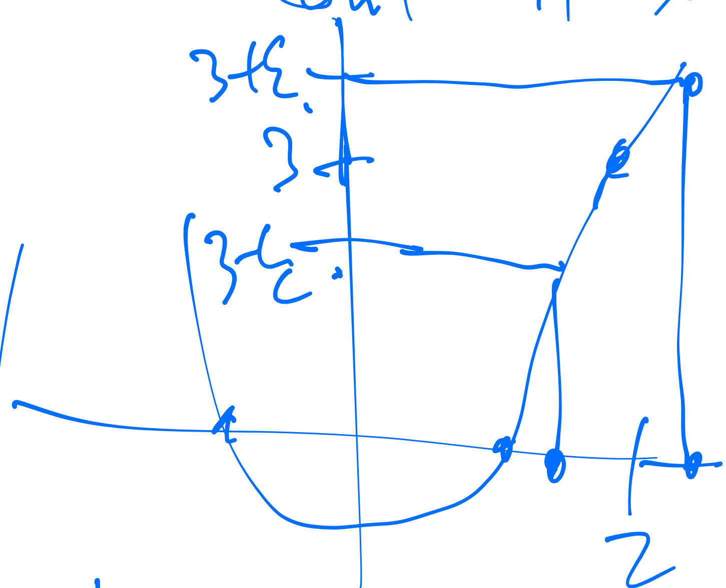
intro x hx

rewrite [show

$x^2-1-(2^2-1)$

$= (x+2)\cdot(x-2)$

by ring_nf],



ε

$\|$

$|x^2-1-3| = |x+2||x-2| < 5\delta$

Note: if $1 \le x \le 3$ then

$|x+2| \le 5$

$|f(x)-L| = \underline{|x-c|}\,\underbrace{|g(x)|}_{\le M} \le \delta\cdot M = \varepsilon.$

hx

$\forall x,\ |x-c| < \delta \to |f x - f c| < \varepsilon.$

rewrite [show

Goal: $|x^2-1-(2^2-1)| < \varepsilon.$

$|(x+2)(x-2)|$

$= |x+2| \cdot |x-2|$ by { done }.

$|x+2| \cdot |x-2| < \varepsilon.$

have f1: $|x-2| \le 1$:= by done

$|(x-2)+4|$

$\le |x-2| + 4.$

$\underbrace{|x-2|}_{<1}$

have f2: $|x-2| \le \varepsilon/5$:= by done.

have f3: $|x+2| \le 5$:= by

rewrite [show $x+2 = x-2+4$ by ring_nf]

have f3': $|x-2+4| \le |x-2| + |4|$:= by apply abs_add

linarith [f3', f1].

have fy: $|x+2| \cdot |x-2| \le 5 \cdot \varepsilon/(5 \cdot \ldots)$ by done

linarith [fy].

Thm ContFn At Add

$f\ g : \mathbb{R} \to \mathbb{R}$ hf : FunContAt f c

$c : \mathbb{R}$ hg : FunContAt g c

Goal: FunContAt $(x \mapsto f\,x + g\,x)$ c.

Let $\varepsilon > 0$ be given.

By hf, we have δ_1 so that

$\forall |x-c| < \delta_1$, $|f\,x - f\,c| < \varepsilon/2$.

Similarly, choose δ_2 using hg

so that $|x-c| < \delta_2 \to |g\,x - g\,c| < \varepsilon/2$.

Use $\delta = \min \delta_1, \delta_2$. Then if $|x-c|<\delta$, we need to show that

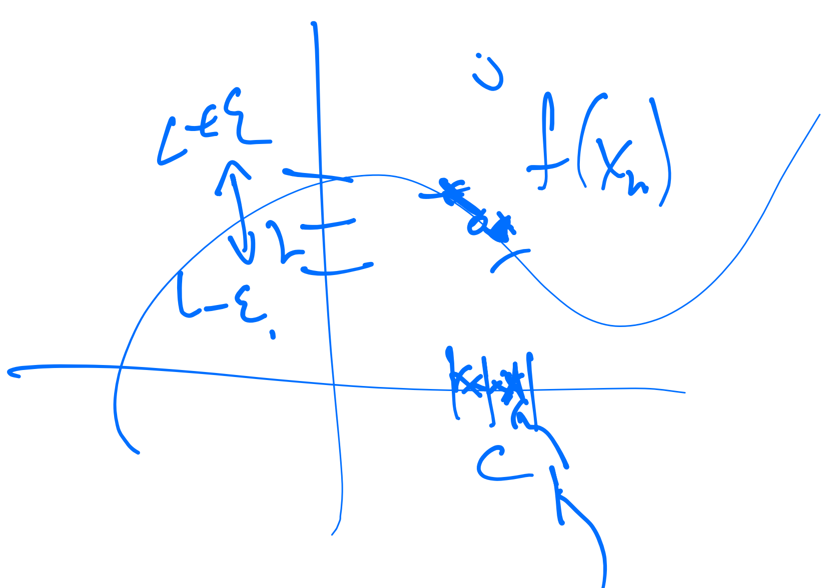
$$|fx+gx-(fc+gc)|<\varepsilon.$$

We compute: $|fx+gx-(fc+gc)|$

$$\le |fx-fc|+|gx-gc| < \varepsilon/2+\varepsilon/2=\varepsilon.$$

Thm: $f \to L$ at c

$\iff \forall x_n \to c$ $(x_n \ne c)$, $f(x_n) \to L$.




c+δ
c
c−δ
x_n
x_n
N
n