An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 19: The Rearrangement Theorems

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Level 1: More Flexible Cauchy

In this level, we strengthen our understanding of absolute convergence by proving a more flexible version of the Cauchy criterion. This will be essential for proving the Rearrangement Theorem.

The Theorem

Theorem (`StrongCauchy_of_AbsSeriesConv`): If a series converges absolutely, then for any $\varepsilon > 0$, there exists N such that for *any* finite set $S \subseteq \mathbb{N}$ whose elements all exceed N, we have:

$$\sum_{k \in S} |a_k| < \varepsilon$$

This is stronger than the usual Cauchy criterion, which only considers consecutive intervals $[n, m)$.

Proof Strategy

The key insight is that if the series of absolute values is Cauchy, then the tail of the series becomes arbitrarily small. Since all elements of S are beyond some point N, the sum over S is bounded by the sum over a larger interval $[N, M)$ where M is chosen large enough to contain all elements of S.

The proof uses:

- The Cauchy criterion for the absolutely convergent series
- The fact that finite sets are bounded, so we can find M containing all elements of S
- Monotonicity of sums of nonnegative terms: if $S \subseteq T$, then $\sum_{k \in S} |a_k| \leq \sum_{k \in T} |a_k|$

New Theorems

- `sum_le_sum_of_nonneg`: If $S \subseteq T$ and $0 \leq f(i)$ for all $i \in T$, then $\sum_{i \in S} f(i) \leq \sum_{i \in T} f(i)$
- `sum_le_mem_of_nonneg`: If $x \in S$ and $0 \leq f(i)$ for all $i \in S$, then $f(x) \leq \sum_{i \in S} f(i)$
- `mem_Ico`: For a and b, we have $x \in [a, b) \leftrightarrow a \leq x \wedge x < b$

The Formal Proof

```
Statement StrongCauchy_of_AbsSeriesConv
    {a : ℕ → ℝ} (ha : AbsSeriesConv a) {ε : ℝ} (hε : ε >
        0) :
    ∃ N, ∀ (S : Finset ℕ), (∀ k ∈ S, k ≥ N) → ∑ k ∈ S, |
       a k| < ε := by
  choose M hM using IsCauchy_of_SeqConv ha ε hε
  use M
  intro S hS
  let sMax := 1 + ∑ k ∈ S, k
  have sMaxIs : sMax = 1 + ∑ k ∈ S, k := by rfl
  have kInS : ∀ k ∈ S, k < sMax := by
    intro n hn
```

```
    have f : id n ≤ ∑ k ∈ S, id k := by
      apply sum_le_mem_of_nonneg hn (by intro; bound)
    change n ≤ ∑ k ∈ S, k at f
    linarith [f, sMaxIs]
  by_cases hSne : S.Nonempty
  choose k0 hk0 using hSne
  have hk0' : M ≤ k0 := by apply hS k0 hk0
  have hk0'' : k0 ≤ sMax := by linarith [kInS k0 hk0]
  have sMaxBnd : M ≤ sMax := by linarith [hk0', hk0'']
  specialize hM M (by bound) sMax (by bound)
  rewrite [show Series (fun n => |a n|) sMax - Series (
     fun n => |a n|) M =
    ∑ n ∈ Ico M sMax, (|a n|) by apply DiffOfSeries _
      sMaxBnd] at hM
  have f0 : 0 ≤ ∑ k ∈ S, |a k| := by
    apply sum_nonneg; intro n hn; bound
  have hM0 : 0 ≤ ∑ k ∈ Ico M sMax, |a k| := by
    apply sum_nonneg; intro k hk; bound
  rewrite [show |∑ k ∈ Ico M sMax, (|a k|)| =
    ∑ k ∈ Ico M sMax, |a k| by apply abs_of_nonneg hM0]
      at hM
  have Ssub : S ⊆ Ico M sMax := by
    intro k hk
    rewrite [mem_Ico]
    split_ands
    apply hS k hk
    linarith [kInS k hk]
  have f2 : ∑ k ∈ S, |a k| ≤ ∑ k ∈ Ico M sMax, |a k| :=
     by
    apply sum_le_sum_of_nonneg Ssub
    intro k hk
    bound
  linarith [f2, hM]
  norm_num at hSne
  rewrite [hSne]
  norm_num
  apply hε
```

Understanding the Proof

Step 1: Use the Cauchy criterion for the absolutely convergent series to get M.

Step 2: Define sMax $= 1 + \sum_{k \in S} k$ as an upper bound for all elements in S.

Step 3: Show that all elements $k \in S$ satisfy $M \leq k <$ sMax, so $S \subseteq [M, \text{sMax})$.

Step 4: By the Cauchy criterion, $\sum_{k \in [M, \text{sMax})} |a_k| < \varepsilon$.

Step 5: Since $S \subseteq [M, \text{sMax})$ and all terms are nonnegative, by `sum_le_sum_of_nonneg`:

$$\sum_{k \in S} |a_k| \leq \sum_{k \in [M, \text{sMax})} |a_k| < \varepsilon$$

Step 6: Handle the edge case where S is empty (then the sum is $0 < \varepsilon$). □

Why This Matters

This theorem is crucial for understanding rearrangements. It tells us that for an absolutely convergent series, not just consecutive terms, but *any collection* of sufficiently large-index terms has arbitrarily small sum. This is what allows us to rearrange terms without affecting convergence.

Level 2: Rearrangements

In this level, we introduce the concept of a rearrangement and prove a key technical lemma about how rearrangements eventually cover all terms.

New Definitions

Definition (`Injective`): A function $f : X \to Y$ is called *injective* (or one-to-one) if for all i, j, we have $f(i) = f(j) \Rightarrow i = j$.

Definition (`Surjective`): A function $f : X \to Y$ is called *surjective* (or onto) if for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

Definition (`Rearrangement`): A function $\sigma : \mathbb{N} \to \mathbb{N}$ is called a *rearrangement* if it is both injective and surjective (i.e., a bijection).

```
def Rearrangement (σ : ℕ → ℕ) : Prop := Injective σ ∧
  Surjective σ
```

A rearrangement σ permutes the natural numbers: it gives us a new ordering of $\mathbb{N}$. For a sequence a, the rearranged sequence $a \circ \sigma$ is defined by $(a \circ \sigma)(n) = a(\sigma(n))$.

The Theorem

Theorem (`EventuallyCovers_of_Rearrangement`): If σ is a rearrangement, then for any $M \in \mathbb{N}$, there exists N such that for all $n \geq N$, the image of σ on $\{0, 1, \ldots, n-1\}$ contains $\{0, 1, \ldots, M-1\}$.

In other words:

$$\text{range}(M) \subseteq \sigma(\text{range}(n)) \text{ for all } n \geq N$$

Proof Strategy

Since σ is surjective, every element $j < M$ appears as $\sigma(k_j)$ for some k_j. Let N be larger than all these k_j values. Then for any $n \geq N$, all the required preimages are in range(n), so their images cover range(M).

The proof uses the axiom of choice (via `choose`) to select the preimages simultaneously.

The Formal Proof

```
Statement EventuallyCovers_of_Rearrangement
    {σ : ℕ → ℕ} (hσ : Rearrangement σ) (M : ℕ) :
    ∃ N, ∀ n ≥ N, (range M) ⊆ image σ (range n) := by
  have surj : ∀ j, ∃ n, σ n = j := hσ.2
  choose τ hτ using surj
  let N := 1 + ∑ k ∈ range M, τ k
  have hN : N = 1 + ∑ k ∈ range M, τ k := by rfl
  use N
  intro n hn m hm
  rewrite [mem_image]
  use τ m
  split_ands
  rewrite [mem_range]
  have hτ' : ∀ k ∈ range M, 0 ≤ τ k := by intro k hk;
     bound
  have f : τ m ≤ ∑ k ∈ range M, τ k := by
    apply sum_le_mem_of_nonneg hm hτ'
  linarith [f, hN, hn]
  apply hτ m
```

Understanding the Proof

Step 1: Since σ is surjective, for each j there exists n such that $\sigma(n) = j$. We use `choose` to get a right inverse function τ such that $\sigma(\tau(j)) = j$ for all j.

Step 2: Define $N = 1+\sum_{k=0}^{M-1} \tau(k)$. This is larger than all the preimages $\tau(0), \tau(1), \ldots, \tau(M-1)$.

Step 3: For any $n \geq N$ and any $m < M$, we have $\tau(m) < N \leq n$, so $\tau(m) \in \text{range}(n)$.

Step 4: Since $\sigma(\tau(m)) = m$, we have $m \in \sigma(\text{range}(n))$.

Therefore, $\text{range}(M) \subseteq \sigma(\text{range}(n))$. □

Intuition

This theorem says that a rearrangement must "eventually catch up" with the original ordering. Even though σ might scramble the order dramatically at first, if we go far enough along (past N), we're guaranteed to have seen all the first M elements.

This is essential for proving that rearranged series behave well: we need to know that we eventually capture all the early terms.

Level 3 – Big Boss: Rearrangement Theorem

This is the culmination of our study: if a series converges absolutely, then any rearrangement of its terms converges to the same sum!

The Fundamental Theorem

Theorem (`RearrangementThm`): If a series $\sum a_n$ converges absolutely, then for any rearrangement $\sigma : \mathbb{N} \to \mathbb{N}$, the rearranged series $\sum a_{\sigma(n)}$ converges to the same limit.

In symbols: If `AbsSeriesConv a` and σ is a `Rearrangement`, then there exists L such that both `SeriesLim a L` and `SeriesLim (a ∘ σ) L`.

The Deep Meaning

This theorem reveals a profound truth about infinite summation:

Infinite summation is commutative if (and only if - as we'll show next) the series is absolutely convergent.

For absolutely convergent series, we can reorder terms however we like—the sum remains unchanged. This is what we expect from finite addition, and absolute convergence is precisely what's needed to extend this property to infinite sums.

New Theorems

- `Series_image`: For an injective function σ, we have

$$\text{Series}(a \circ \sigma, n) = \sum_{k \in \sigma(\text{range}(n))} a_k$$

- `sum_sdiff`: If $S_1 \subseteq S_2$, then

$$\sum_{x \in S_2 \setminus S_1} f(x) + \sum_{x \in S_1} f(x) = \sum_{x \in S_2} f(x)$$

- `abs_sum_le_sum_abs`: Triangle inequality for finite sums:

$$\left| \sum_{x \in S} f(x) \right| \leq \sum_{x \in S} |f(x)|$$

Proof Strategy

Let L be the limit of the original series $\sum a_n$. We need to show that $\sum a_{\sigma(n)} \to L$ as well.

Given $\varepsilon > 0$:

1. Use the strong Cauchy property (which we just proved in Level 1) to get N_1 such that for any set S with elements $\geq N_1$, we have $\sum_{k \in S} |a_k| < \varepsilon/2$
2. Use the convergence of $\sum a_n$ to L to get N_2 such that $|\text{Series}(a, n) - L| < \varepsilon/2$ for $n \geq N_2$
3. Use `EventuallyCovers_of_Rearrangement` (from Level 2) to get N_3 such that for $n \geq N_3$, we have $\text{range}(N_1 + N_2) \subseteq \sigma(\text{range}(n))$
4. For $n \geq N_1 + N_2 + N_3 + 1$, write:
$$\text{Series}(a \circ \sigma, n) - L = [\text{Series}(a \circ \sigma, n) - \text{Series}(a, N_1 + N_2)] + [\text{Series}(a, N_1 + N_2) - L]$$
5. The second term is $< \varepsilon/2$ by step 1
6. For the first term, use `Series_image` to rewrite it as a sum over $\sigma(\text{range}(n)) \setminus \text{range}(N_1 + N_2)$
7. By step 3, the elements in this set difference are all $\geq N_1 + N_2$
8. By the strong Cauchy property (step 2), this sum is $< \varepsilon/2$
9. Therefore $|\text{Series}(a \circ \sigma, n) - L| < \varepsilon$

The Formal Proof

```
Statement RearrangementThm {a : ℕ → ℝ} (ha :
   AbsSeriesConv a) {σ : ℕ → ℕ}
  (hσ : Rearrangement σ) : ∃ L, SeriesLim a L ∧
     SeriesLim (a ∘ σ) L := by
  choose L hL using Conv_of_AbsSeriesConv ha
  use L
  split_ands
  apply hL
  intro ε hε
```

```
apply IsCauchy_of_SeqConv at ha
choose N1 hN1 using ha (ε / 2) (by bound)
choose N2 hN2 using hL (ε / 2) (by bound)
choose N3 hN3 using EventuallyCovers_of_Rearrangement
   hσ (N1 + N2)
use N1 + N2 + N3 + 1
intro n hn
specialize hN2 (N1 + N2) (by bound)
specialize hN1 (N1 + N2) (by bound)
rewrite [show Series (a ∘ σ) n - L =
  (Series (a ∘ σ) n - Series a (N1 + N2)) +
  (Series a (N1 + N2) - L) by ring_nf]
have f1 : |Series (a ∘ σ) n - Series a (N1 + N2) +
            (Series a (N1 + N2) - L)|
        ≤ |Series (a ∘ σ) n - Series a (N1 + N2)| +
          |(Series a (N1 + N2) - L)| := by apply
             abs_add
have f2 : |Series (a ∘ σ) n - Series a (N1 + N2)| =
    |∑ k ∈ image σ (range n) \ range (N1 + N2), a k|
       := by
  have f : Series (a ∘ σ) n = ∑ k ∈ image σ (range n),
      a k :=
    Series_image a σ hσ.1 n
  rewrite [f]
  change |(∑ k ∈ image σ (range n), a k) -
          (∑ k ∈ range (N1 + N2), a k)| =
         |(∑ k ∈ image σ (range n) \ range (N1 + N2),
            a k)|
  rewrite [← Finset.sum_sdiff (hN3 n (by bound))]
  ring_nf
have f3 : |∑ k ∈ image σ (range n) \ range (N1 + N2),
   a k|
    ≤ ∑ k ∈ image σ (range n) \ range (N1 + N2), |a k|
        := by
  apply abs_sum_le_sum_abs
let M := N1 + N2 + 1 + ∑ k ∈ range n, σ k
have Mis : M = N1 + N2 + 1 + ∑ k ∈ range n, σ k := by
   rfl
have Mbnd : ∀ k ∈ range n, σ k < M := by
  intro k hk
```

```
    rewrite [Mis]
    have f : σ k ≤ ∑ j ∈ range n, σ j := by
      apply sum_le_mem_of_nonneg hk; intro i hi; bound
    linarith [f]
  have f4 : ∑ k ∈ image σ (range n) \ range (N1 + N2), |
     a k| ≤
          ∑ k ∈ Ico (N1 + N2) M, |a k| := by
    apply sum_le_sum_of_nonneg
    intro i hi
    rewrite [mem_Ico]
    rewrite [mem_sdiff] at hi
    rewrite [mem_range] at hi
    rewrite [mem_image] at hi
    have hi2 : N1 + N2 ≤ i := by bound
    split_ands
    apply hi2
    have hi1 : ∃ a ∈ range n, σ a = i := by apply hi.1
    choose a ha using hi1
    rewrite [← ha.2]
    apply Mbnd a ha.1
    intro i hi;
    bound
  have sNonneg : 0 ≤ ∑ k ∈ range n, σ k := by
    apply sum_nonneg
    intro i hi
    bound
  have Mbnd : N1 + N2 ≤ M := by rewrite [Mis]; linarith
     [sNonneg]
  specialize hN1 M Mbnd
  rewrite [DiffOfSeries _ Mbnd] at hN1
  have f5 : ∑ k ∈ Ico (N1 + N2) M, |a k| ≤
           |∑ k ∈ Ico (N1 + N2) M, (|a k|)| := by bound
  linarith [f1, hN2, f2, f3, f4, f5, hN1, hN3]
```

Understanding the Proof

The proof is intricate but follows the strategy outlined above. The key steps are:

Step 1-3: Obtain three thresholds N_1, N_2, N_3 from the three key properties (strong Cauchy, convergence, eventual covering).

Step 4-5: Decompose the difference $\text{Series}(a \circ \sigma, n) - L$ into two parts and apply the triangle inequality.

Step 6-8: Use `Series_image` and `sum_sdiff` to rewrite the first part as a sum over elements that haven't been "covered" yet by the rearrangement.

Step 9-12: Show this uncovered set consists only of large indices, then apply the strong Cauchy property to bound it.

Step 13: Combine all inequalities using `linarith` to get $|\text{Series}(a \circ \sigma, n) - L| < \varepsilon$. $\square$

Historical Note

This theorem was known to Augustin-Louis Cauchy and Niels Henrik Abel in the 1820s, though earlier mathematicians like Dirichlet had worked with similar ideas. It's a cornerstone of rigorous analysis and marks a sharp distinction between finite and infinite operations.

The proof is intricate but beautiful, combining all the machinery we've developed: Cauchy sequences, absolute convergence, the strong Cauchy property, and the covering property of rearrangements.

Level 4 – Bigger Boss: Conditional Convergence Theorem

We now arrive at one of the most surprising and dramatic results in all of real analysis: Riemann's Rearrangement Theorem.

The Shocking Theorem

Theorem (Riemann's Rearrangement Theorem): If a series $\sum a_n$ converges but does *not* converge absolutely (i.e., it is *conditionally convergent*), then for **any** real number L, there exists a rearrangement σ such that the rearranged series $\sum a_{\sigma(n)}$ converges to L.

In fact (though we won't prove this part), there also exist rearrangements that diverge to $+\infty$, diverge to $-\infty$, or oscillate without converging at all!

Statement

```
Statement {a : ℕ → ℝ} (ha1 : SeriesConv a) (ha2 : ¬
    AbsSeriesConv a) :
  ∀ L, ∃ (σ : ℕ → ℕ) (hσ : Rearrangement σ), SeriesLim (
      a ∘ σ) L := by
  sorry
```

What This Means

This theorem tells us something profound:

If a series is only conditionally convergent, then infinite summation is as non-commutative as possible!

By cleverly rearranging the terms, we can make the series converge to *literally any* value we want. The sum we get depends entirely on the order in which we add the terms.

A Concrete Example

Consider the alternating harmonic series:

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2$$

This series converges to $\ln 2 \approx 0.693$, but it does not converge absolutely (since the harmonic series diverges).

By Riemann's theorem, we can rearrange its terms to make it converge to:

- π (or any other positive number)
- 0 (or any negative number)
- 1000000 (or any huge number)
- $-1/137$ (or any specific target)

Example construction to get $\pi/4$:
Take two positive terms, then one negative:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

With the right pattern of positive and negative terms, this rearrangement converges to $\pi/4$.

Proof Sketch

The proof (which we leave as `sorry`) uses a greedy algorithm:

To make the series converge to L:

1. Separate the positive and negative terms: $P = \{a_n : a_n > 0\}$ and $N = \{a_n : a_n < 0\}$
2. Since the series converges, $a_n \to 0$, so both positive and negative terms go to zero
3. Since the series doesn't converge absolutely, both $\sum P$ and $\sum N$ diverge (this requires proof!)
4. Add positive terms until the partial sum just exceeds L
5. Add negative terms until the partial sum just drops below L
6. Repeat: alternately add positive and negative terms to stay close to L

7. Since the terms go to zero, the oscillations become arbitrarily small, and the series converges to L

The formal proof requires careful bookkeeping to define the rearrangement σ and show it has the desired properties.

To make a conditionally convergent series $\sum a_n$ diverge to $+\infty$ by rearrangement, note that we cannot simply take all the positive terms and ignore the negative terms—that wouldn't be a rearrangement at all, since a rearrangement must include every term of the original series exactly once. Instead, use your favorite a greedy algorithm that carefully interleaves positive and negative terms. For example, start by adding positive terms $a_{p_1}, a_{p_2}, a_{p_3}, \ldots$ until the partial sum exceeds 1. Then add just enough negative terms $a_{n_1}, a_{n_2}, \ldots$ to bring the sum back down, say, to 0. Next, add positive terms until the sum exceeds 2, then add negative terms to keep the sum above 1. Continue this pattern: add positive terms until the sum exceeds k, then add negative terms to keep it above $k-1$. Since both the positive and negative series diverge (a key property of conditional convergence), we never run out of terms of either sign. Moreover, since $a_n \to 0$, each "correction" with negative terms requires only finitely many terms, and we can ensure every term appears exactly once in this rearrangement. The partial sums form a sawtooth pattern that marches steadily upward, with the lower bound increasing by 1 at each stage, guaranteeing divergence to $+\infty$. By sprinkling in the negative terms strategically, we satisfy the requirement that all terms appear while still achieving divergence.

Why Both Series Diverge

A key lemma (not proven here): If $\sum a_n$ converges but $\sum |a_n|$ diverges, then both the series of positive terms and the series of negative terms must diverge.

Proof idea: If $\sum a_n^+ < \infty$ (where $a_n^+ = \max(a_n, 0)$), then since $a_n^- = a_n^+ - a_n$ and $\sum a_n < \infty$, we would have $\sum a_n^- < \infty$, hence $\sum |a_n| = \sum (a_n^+ + a_n^-) < \infty$, contradiction.

Philosophical Implications

This theorem reveals a fundamental difference between:

- **Absolutely convergent series**: Behave like finite sums—rearrangements don't matter
- **Conditionally convergent series**: Delicate balances where order matters—rearrangements change everything

It's a powerful reminder that intuitions from finite mathematics don't always extend to the infinite realm. Infinite series are more subtle than they appear!

Historical Note

This theorem was discovered by Bernhard Riemann in 1854 during his Habilitation at the University of Göttingen. It was part of his groundbreaking work on trigonometric series and helped establish modern standards of rigor in analysis.

The result shocked the mathematical community and demonstrated that convergence alone is not enough—*how* a series converges (absolutely vs. conditionally) profoundly affects its properties.

Peter Gustav Lejeune Dirichlet had already observed special cases, but Riemann gave the complete general theorem. The proof was one of many contributions in Riemann's dissertation that revolutionized analysis.

Conclusion

With this theorem, we've reached the pinnacle of our study of sequences and series (before we get to functions). We've seen that:

- Absolute convergence $\Rightarrow$ rearrangement invariance (Level 3)
- Conditional convergence $\Rightarrow$ complete rearrangement chaos (Level 4)

These two theorems together give us a complete dichotomy: a series either has rearrangement-invariant sum (absolute convergence) or its sum can be anything we want (conditional convergence). There's no middle ground!

This is one of the deepest and most beautiful results in real analysis, revealing the profound difference between absolute and conditional convergence.

Last time: saw that

Problem: Infinite summation is not commutative.

- Abs Conv $\Rightarrow$ Conv. $\sum_{i=0}^{\infty} |a_i| < \infty \Rightarrow \sum a_i$ Conv.
- Alternating Series Test: $a_i \searrow 0$. Then $\sum (-1)^n a_n$ Conv.

Thm: If series $\sum a_n$ is Abs Conv $\Rightarrow$ inf summation is commutative.

Thm: If not, then inf summation is as non-commutative as possible!!!

(Riemann Rearrangement Thm): For any target $L \in [-\infty, \infty]$ $\exists$ rearrangement that sums to L.

Def: A rearrangement $\sigma : \mathbb{N} \to \mathbb{N}$ that "shuffles" $\mathbb{N}$.

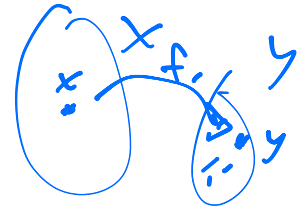
$$\sum_{n=0}^{\infty} a_{\sigma(n)}$$

- hits every number "surjection"
- never repeats a number. "injection".

rearrangement = "bijection".

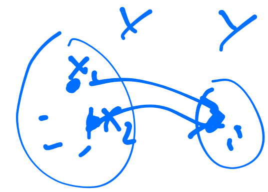
Def: $f : X \to Y$ is surjective if:

$$\forall y : Y,\ \exists x : X,\ f(x) = y.$$



Def: $f : X \to Y$ is injective if:

$$\forall x_1, x_2 : X,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$



Thm 1: **Objects:** $a : \mathbb{N} \to \mathbb{R}$,

Assumptions: ha: AbsSeriesConv a. $\left(\sum |a_n| < \infty\right)$,

Goal: $\exists$ L, SeriesLim a L $\wedge$ $\forall \sigma$,

Rearrangement σ $\to$ SeriesLim $(a \circ \sigma)$ L.

- choose L hL using Conv_of_AbsSeriesConv ha

use L

split_ands

apply hL.

intro σ hσ.

hL: SeriesLim a L

(SeqLim (Series a) L)

hσ: Injective σ $\wedge$ Surjective σ

Sketch: Idea:

need to take n large so that: $\left|\sum_{k \in \text{range } n} a_{\sigma_k} - L\right| < \varepsilon.$

$\forall L \rightsquigarrow$ $\left| \sum_{k \in \text{range } \sigma n} a_k - L \right| < \varepsilon.$

m large

<u>Idea:</u> image σ (range n) $= \{ \sigma(k) \mid k \in \text{range } n \}$

$\forall m, \exists n,$ $\underbrace{\text{range } m}$ $\subseteq$

<u>Note:</u> Our matrix row vs column summation is <u>Not</u> a rearrangement.

<u>Idea:</u> $\le \underbrace{\left| \sum_{k=0}^{n-1} a_{\sigma_k} - \sum_{k=0}^{m} a_k \right|}$ $+ \underbrace{\left| \sum_{k=0}^{m} a_k - L \right|}_{< \varepsilon/2.}$

image σ (range n)

range m

$= \left| \sum_{k \in S} a_k \right| \le \sum_{k \in S} |a_k| \le \sum_{k \in I \text{ to } \infty M} |a_k| < \varepsilon/2$ by Cauchy lol.

where S is some finite set

Again: $\sum_{k=0}^{m-1} |a_k|$ is Cauchy $\Rightarrow \left| \underbrace{\sum^{m-1} |a_k| - \sum^{n-1} |a_k|}_{\sum_{I_{co}\, n\, m} |a_k|} \right| < \varepsilon$,

$\forall \varepsilon > 0\ \exists N, \forall n \geq N\ \forall m \geq n$.

<u>Thm</u> (Strong Cauchy_of_AbsSeriesConv)

$a : \mathbb{N} \to \mathbb{R}$ ha : AbsSeriesConv a,

$\varepsilon : \mathbb{R}$ hε : $\varepsilon > 0$.

<u>Goal</u>: $\exists N, \forall S$: Finset N,

$(\forall k \in S, k \geq N) \to \sum_{k \in S} |a_k| < \varepsilon$.

<u>Proof</u>: choose N hN using IsCauchy_of_SeqConv ha

use N intro S hS

$\longrightarrow$ Let $M := \max_{k \in S} k$.

$\sum_{k \in S} |a_k| \leq \sum_{k \in I_{co}\, N\, M} |a_k| < \varepsilon$.

Thm 2 (Eventually Covers of Rearrangement)

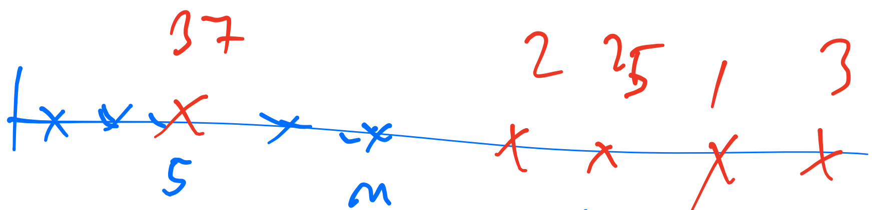
$\sigma : \mathbb{N} \to \mathbb{N}$ $\quad h\sigma$: Rearrangement σ.

$M : \mathbb{N}$.

Goal: $\exists N, \forall n \geq N, \underbrace{\text{range } M \subseteq \text{image } \sigma \, (\text{range } n)}$

$\Updownarrow$

$\forall k, \; k \in \text{range } M \to k \in \text{image } \sigma \, (\text{range } n)$



Choose τ $h\tau$ using $h\sigma.2 : \forall m, \exists n, \sigma(n) = m$.

$\tau : \mathbb{N} \to \mathbb{N}$

$h\tau : \forall m, \; \sigma(\tau \, m) = m$.

Let $N := 1 + \sum_{k \in \text{range } M} \tau(k)$ ($\cancel{\max}$)

Use N.

intro n hn $\leftarrow$ $hn : n \geq N$

intro k hk $\leftarrow$ $hk : \underline{k \in \text{range } M}$.

$\Updownarrow$

$k < M$.

rewrite [mem_image]

$\boxed{\text{mem\_Ico, mem\_range, mem\_image}}$

Use $\tau(k)$.

New Goal: $\exists j \in \text{range } n, \; \sigma(j) = k$.

Thm 4: Conditional Convergence Thm:

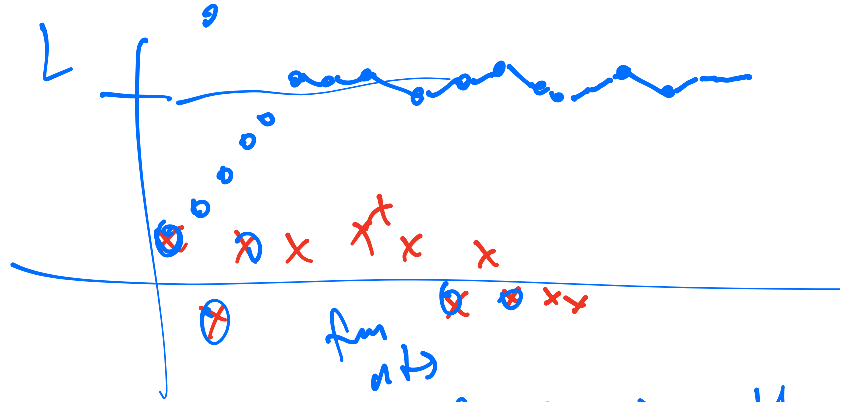
$a : \mathbb{N} \to \mathbb{R}$ $\quad ha1$: Series Conv a

$ha2 : \neg$ Abs Series Conv a

$\Big\}$ Conditionally Converges a.

Goal: $\forall L : \mathbb{R}, \exists \sigma$, Rearrangement σ ∧ Series Lim $(a \circ \sigma)$ L,

Idea:



$a_n \to 0$, $|a| = a_+ + a_-$

let aPos : $\mathbb{N} \to \mathbb{R}$:= if $a_n \geq 0$ then a_n else 0,

$a = a_+ - a_-$

aNeg := max $-a$ 0,

haPos : SeriesTendsToInfty aPos

haNeg : Series — — aNeg

$$\begin{cases} \sum a_+ n = \infty \\ \sum a_- n = \infty \end{cases}$$

Idea:

	$+ = \infty$	$+ < \infty$
$a_- = \infty$	✓.	¬ Series Lim
$- < \infty$	¬ Series Lim	Abs Lim X

So $\forall N$, $\forall x > 0$, $\exists m \geq N$ s.t. $\sum_{k \in \text{Ico } N\ m} a_+ k > x$,

For $L = \infty$ do Pos