An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 17: Series and Convergence

Prof. Alex Kontorovich

This text is automatically generated by LLM from "Real Analysis, The Game", Lecture 17

1 Level 1 (of Lecture 16): The Vanishing Term Test

In this foundational level, we introduce the concept of infinite series and prove a fundamental necessary condition for convergence.

1.1 Defining Series

Given a sequence $a : \mathbb{N} \to \mathbb{R}$, we define a new sequence called the **series** of a, which consists of the partial sums:

Definition (Series): For a sequence $a : \mathbb{N} \to \mathbb{R}$, we define:

$$\text{Series}(a)(n) = \sum_{k=0}^{n-1} a_k$$

That is, the n-th term of $\text{Series}(a)$ is the sum of the first n terms of a.

For example, if $a = (1, 2, 3, 4, \ldots)$, then:

$$\begin{aligned}
\text{Series}(a)(0) &= 0 \\
\text{Series}(a)(1) &= 1 \\
\text{Series}(a)(2) &= 1 + 2 = 3 \\
\text{Series}(a)(3) &= 1 + 2 + 3 = 6 \\
\text{Series}(a)(4) &= 1 + 2 + 3 + 4 = 10
\end{aligned}$$

1.2 Series Convergence

A **series converges** if its sequence of partial sums converges.

Definition (SeriesConv): We say $\text{SeriesConv}(a)$ holds if $\text{SeqConv}(\text{Series}(a))$ holds. That is, there exists a limit $L \in \mathbb{R}$ such that:

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} a_k = L$$

We write this as $\sum_{k=0}^{\infty} a_k = L$ and say the infinite series converges to L.

Definition (SeriesLim): If the series of a converges to L, we write $\text{SeriesLim}(a, L)$, which means $\text{SeqLim}(\text{Series}(a), L)$.

1.3 The Vanishing Term Test

Now we prove a crucial necessary condition for convergence:

Theorem (LimZero_of_SeriesConv): If a series $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{n\to\infty} a_n = 0$.

In other words: *if the terms of a series don't go to zero, the series cannot converge.*

1.4 Proof Strategy

The key insight is that if a series converges, then its sequence of partial sums is convergent, hence Cauchy. For a Cauchy sequence, consecutive terms get arbitrarily close together.

Now, the difference between consecutive partial sums is exactly one term of the original sequence:

$$\text{Series}(a)(n+1) - \text{Series}(a)(n) = \sum_{k=0}^{n} a_k - \sum_{k=0}^{n-1} a_k = a_n$$

So if the partial sums are Cauchy, then a_n must approach zero.

1.5 The Formal Proof

Proof: Let $\varepsilon > 0$ be given. We need to find N such that for all $n \geq N$, we have $|a_n - 0| < \varepsilon$.

Since the series converges, Series(a) converges, and therefore Series(a) is Cauchy. By the definition of Cauchy sequence, there exists N such that for all $m, n \geq N$ with $m \geq n$:

$$|\text{Series}(a)(m) - \text{Series}(a)(n)| < \varepsilon$$

Now, for any $n \geq N$, choose $m = n + 1$. Then:

$$\begin{aligned} |a_n| &= |a_n - 0| \\ &= \left| \sum_{k=0}^{n} a_k - \sum_{k=0}^{n-1} a_k \right| \\ &= |\text{Series}(a)(n+1) - \text{Series}(a)(n)| \\ &< \varepsilon \end{aligned}$$

This completes the proof. □

1.6 The Lean Proof

```
Statement LimZero_of_SeriesConv (a : ℕ → ℝ)
    (ha : SeriesConv a) : SeqLim a 0 := by
  intro ε hε
  change SeqConv (Series a) at ha
  have cau : IsCauchy (Series a) := by
    apply IsCauchyOfLim (Series a) ha
  choose N hN using cau ε hε
  use N
  intro n hn
  specialize hN n hn (n+1) (by bound)
  change |∑ k ∈ range (n+1), a k - ∑ k ∈ range n, a k| <
      ε at hN
  rewrite [show ∑ k ∈ range (n+1), a k =
```

```
    ∑ k ∈ range n, a k + a n by apply sum_range_succ] at
        hN
  rewrite [show ∑ k ∈ range n, a k + a n -
    ∑ k ∈ range n, a k = a n by ring_nf] at hN
  rewrite [show a n - 0 = a n by ring_nf]
  apply hN
```

1.7 Understanding the Theorem

This theorem gives us a quick **divergence test**: if we want to show that a series $\sum a_k$ diverges, it suffices to show that a_k does not approach zero.

Examples of divergent series:

- $\sum_{k=0}^{\infty} 1 = 1 + 1 + 1 + \cdots$ diverges because the terms don't go to zero
- $\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \cdots$ diverges because the terms oscillate between 1 and -1
- $\sum_{k=1}^{\infty} \frac{k}{k+1}$ diverges because $\frac{k}{k+1} \to 1 \neq 0$

1.8 Warning: The Converse is False!

It is **not** true that if $a_n \to 0$, then $\sum a_k$ converges. The terms going to zero is *necessary* but *not sufficient* for convergence.

The classic counterexample is the **harmonic series**:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Here $\frac{1}{k} \to 0$, but the series diverges! The partial sums grow like $\log n$, going to infinity.

To prove convergence, we need additional tests (comparison test, ratio test, integral test, etc.), which we will develop in subsequent levels.

1.9 Historical Note

This theorem is sometimes called the **n-th term test for divergence** or the **vanishing term test**. It was known to early analysts like Nicole Oresme

(14th century) and Jakob Bernoulli (17th century), who used it to identify divergent series quickly.

The harmonic series, which shows the converse fails, was proven to diverge by Oresme around 1350 using a clever grouping argument.

2 Level 1: Leibniz Series – Partial Sums

In this level, we begin our study of infinite series by examining a beautiful classical result discovered by Leibniz. We will evaluate the series

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots$$

The key to understanding an infinite series is to first understand its **partial sums**. For a sequence $a : \mathbb{N} \to \mathbb{R}$, the n-th partial sum is defined as:

$$\text{Series}(a, n) = \sum_{k=0}^{n-1} a_k$$

Our first task is to find an explicit formula for the partial sums of the Leibniz series.

2.1 The Theorem

Theorem (LeibnizSeries'): Let $a : \mathbb{N} \to \mathbb{R}$ be the sequence defined by $a_n = \frac{1}{(n+1)(n+2)}$ for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{n-1} a_k = 1 - \frac{1}{n+1}$$

2.2 Proof Strategy

The key insight is to use **telescoping**: we can rewrite each term $\frac{1}{(k+1)(k+2)}$ using partial fractions as:

$$\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2}$$

When we sum these terms, most cancel out:

$$\begin{aligned}
\sum_{k=0}^{n-1} \frac{1}{(k+1)(k+2)} &= \sum_{k=0}^{n-1} \left(\frac{1}{k+1} - \frac{1}{k+2} \right) \\
&= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= 1 - \frac{1}{n+1}
\end{aligned}$$

2.3 The Formal Proof

We prove this by induction on n.

Base case ($n = 0$): The sum over an empty range is 0, and $1 - \frac{1}{0+1} = 1 - 1 = 0$. (In practice, this requires checking boundary conditions carefully.)

Inductive step: Assume the formula holds for $n = m$. We need to show it holds for $n = m + 1$. We have:

$$\begin{aligned}
\sum_{k=0}^{m} a_k &= \sum_{k=0}^{m-1} a_k + a_m \\
&= \left(1 - \frac{1}{m+1}\right) + \frac{1}{(m+1)(m+2)} \qquad \text{(by inductive hypothesis)} \\
&= 1 - \frac{1}{m+1} + \frac{1}{(m+1)(m+2)}
\end{aligned}$$

Now we simplify:

$$1 - \frac{1}{m+1} + \frac{1}{(m+1)(m+2)} = 1 - \frac{1}{m+2}$$

This completes the proof.

2.4 Understanding the Result

This formula tells us that as $n \to \infty$, the partial sums approach 1:

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{(k+1)(k+2)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

Therefore, the infinite series converges to 1.

2.5 The Lean Proof

The formal proof in Lean uses induction (`induction' n with m hm`) and algebraic simplification tactics:

```
Statement LeibnizSeries' (a : ℕ → ℝ)
    (ha : ∀ n, a n = 1 / ((n + 1) * (n + 2))) :
    ∀ n, ∑ k ∈ range n, a k = 1 - 1 / (n + 1) := by
  intro n
```

```
induction' n with m hm
bound  -- base case
rewrite [show ∑ k ∈ range (m + 1), a k =
  ∑ k ∈ range m, a k + a m by apply sum_range_succ]
rewrite [hm]  -- apply inductive hypothesis
rewrite [ha m]  -- substitute definition of a
push_cast
norm_num
field_simp
ring_nf
```

The tactics `field_simp` and `ring_nf` handle the algebraic manipulation automatically.

3 Level 2: Leibniz Series – Convergence

Having established an explicit formula for the partial sums, we now prove that the Leibniz series actually converges.

3.1 The Definition of Series Convergence

Recall that a series $\sum_{k=0}^{\infty} a_k$ **converges** if its sequence of partial sums converges. That is, `SeriesConv a` means there exists a limit $L \in \mathbb{R}$ such that:

$$\forall \varepsilon > 0, \exists N, \forall n \geq N, \left| \sum_{k=0}^{n-1} a_k - L \right| < \varepsilon$$

3.2 The Theorem

Theorem (LeibnizSeries): Let $a : \mathbb{N} \to \mathbb{R}$ be the sequence defined by $a_n = \frac{1}{(n+1)(n+2)}$ for all $n \in \mathbb{N}$. Then the series $\sum_{k=0}^{\infty} a_k$ converges.

3.3 Proof Strategy

From Level 1, we know that:

$$\sum_{k=0}^{n-1} a_k = 1 - \frac{1}{n+1}$$

To show convergence, we need to prove that this sequence of partial sums converges to $L = 1$. That is, we need to show:

$$\forall \varepsilon > 0, \exists N, \forall n \geq N, \left| \left(1 - \frac{1}{n+1}\right) - 1 \right| < \varepsilon$$

Simplifying the left side:

$$\left| \left(1 - \frac{1}{n+1}\right) - 1 \right| = \left| -\frac{1}{n+1} \right| = \frac{1}{n+1}$$

So we need to show:

$$\forall \varepsilon > 0, \exists N, \forall n \geq N, \frac{1}{n+1} < \varepsilon$$

3.4 Using the Archimedean Property

The key is the **Archimedean Property**: for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$, or equivalently, $N > \frac{1}{\varepsilon}$.

Given $\varepsilon > 0$, choose N such that $\frac{1}{N} < \varepsilon$. Then for all $n \geq N$:

$$\frac{1}{n+1} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

This completes the proof.

3.5 The Formal Proof

The Lean proof follows this structure:

```
Statement LeibnizSeries (a : ℕ → ℝ)
    (ha : ∀ n, a n = 1 / ((n + 1) * (n + 2))) :
    SeriesConv a := by
  have f : ∀ n, ∑ k ∈ range n, a k = 1 - 1 / (n + 1) :=
     by
    apply LeibnizSeries' a ha
  use 1  -- the limit is 1
  intro ε hε
  choose N hN using ArchProp hε  -- get N from
     Archimedean property
  use N
  intro n hn
  change |∑ k ∈ range n, a k - 1| < ε
  rewrite [f n]  -- substitute partial sum formula
  rewrite [show |(1 : ℝ) - 1 / (n + 1) - 1| =
    |-((1 : ℝ) / (n + 1))| by ring_nf]
  rewrite [show |- ((1 : ℝ) / (n + 1))| =
    |(1 : ℝ) / (n + 1)| by apply abs_neg]
  rewrite [show |((1 : ℝ) / (n + 1))| =
    (1 : ℝ) / (n + 1) by apply abs_of_pos (by bound)]
  -- Now we have to show 1/(n+1) < ε
  have hn' : (N : ℝ) ≤ n := by exact_mod_cast hn
  have hn'' : (1 : ℝ) / n ≤ 1 / N := by field_simp;
     bound
  have hN' : (1 : ℝ) / N < ε := by
    field_simp; field_simp at hN; linarith [hN]
```

```
  have hn''' : (1 : ℝ) / (n + 1) ≤ 1 / n := by
    field_simp; bound
  linarith [hn''', hn'', hN']
```

3.6 Understanding the Convergence

The Leibniz series converges to 1, which we can verify:

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots = 1$$

The rate of convergence is $O(1/n)$: after n terms, the partial sum differs from the limit by approximately $\frac{1}{n+1}$.

4 Level 3: Series Order Theorem

One of the fundamental properties of series is that they respect the order of their terms: if we have two sequences where one is term-by-term less than or equal to the other, then the same relationship holds for their partial sums.

4.1 The Theorem

Theorem (SeriesOrderThm): Let $a, b : \mathbb{N} \to \mathbb{R}$ be two sequences such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$:

$$\text{Series}(a, n) \leq \text{Series}(b, n)$$

That is:

$$\sum_{k=0}^{n-1} a_k \leq \sum_{k=0}^{n-1} b_k$$

4.2 Proof Strategy

This is a straightforward induction argument. If each term $a_k \leq b_k$, then adding up the first n terms preserves this inequality.

Base case ($n = 0$): Both sums are empty, so $0 \leq 0$.

Inductive step: Assume $\sum_{k=0}^{n-1} a_k \leq \sum_{k=0}^{n-1} b_k$. We need to show:

$$\sum_{k=0}^{n} a_k \leq \sum_{k=0}^{n} b_k$$

We can write:

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{n-1} a_k + a_n$$

$$\sum_{k=0}^{n} b_k = \sum_{k=0}^{n-1} b_k + b_n$$

By the inductive hypothesis, $\sum_{k=0}^{n-1} a_k \leq \sum_{k=0}^{n-1} b_k$. By assumption, $a_n \leq b_n$. Adding these inequalities:

$$\sum_{k=0}^{n-1} a_k + a_n \leq \sum_{k=0}^{n-1} b_k + b_n$$

This completes the proof.

4.3 The Formal Proof

```
Statement SeriesOrderThm (a b : ℕ → ℝ)
    (hab : ∀ n, a n ≤ b n) :
    ∀ n, Series a n ≤ Series b n := by
  intro n
  induction' n with n hn
  bound  -- base case: 0 ≤ 0
  change ∑ k ∈ range (n + 1), a k ≤ ∑ k ∈ range (n + 1),
      b k
  change ∑ k ∈ range (n), a k ≤ ∑ k ∈ range (n), b k at
     hn
  rewrite [show ∑ k ∈ range (n + 1), a k =
    ∑ k ∈ range n, a k + a n by apply sum_range_succ]
  rewrite [sum_range_succ]
  linarith [hab n, hn]
```

4.4 Applications

This theorem is crucial for comparison tests in series convergence theory. If we know that $\sum b_k$ converges and $0 \leq a_k \leq b_k$, then we can conclude that $\sum a_k$ also converges (and converges to a value at most as large as $\sum b_k$).

4.5 Consequences for Infinite Series

If $a_n \leq b_n$ for all n and both series converge, then:

$$\sum_{k=0}^{\infty} a_k \leq \sum_{k=0}^{\infty} b_k$$

This follows by taking limits of the partial sum inequality.

Example: We will use this theorem in the next level to bound the Basel series by the Leibniz series, showing that the Basel series converges.

5 Level 4: The Basel Problem

We now turn to one of the most famous problems in the history of mathematics: the **Basel Problem**, posed by Pietro Mengoli in 1644 and famously solved by Leonhard Euler in 1734.

5.1 Historical Context

Near the turn of the 18th century, the Bernoulli brothers, Johann and Jakob, became obsessed with evaluating the series:

$$\sum_{k=0}^{\infty} \frac{1}{(k+2)^2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

Despite their considerable efforts, they could not find its exact value. It would take their most famous pupil, Leonhard Euler, to solve it in 1734, showing that:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

(Our series starts at $k = 2$, so it equals $\frac{\pi^2}{6} - 1$.)

In this level, we will prove something more modest: that the series converges at all.

5.2 The Strategy: Comparison with the Leibniz Series

The key insight is to compare our series with the Leibniz series from earlier. We have:

$$\frac{1}{(k+2)^2} = \frac{1}{(k+2)(k+2)} \leq \frac{1}{(k+1)(k+2)}$$

Since the denominators satisfy $(k+2)^2 = (k+2)(k+2) \geq (k+1)(k+2)$ for all $k \geq 0$, the inequality holds.

By the Series Order Theorem (Level 3), this means:

$$\sum_{k=0}^{n-1} \frac{1}{(k+2)^2} \leq \sum_{k=0}^{n-1} \frac{1}{(k+1)(k+2)}$$

From Level 1, we know the right side equals $1 - \frac{1}{n+1} < 1$. Therefore, the partial sums of the Basel series are bounded above by 1.

5.3 Using the Monotone Bounded Convergence Theorem

The partial sums of the Basel series form a **monotone increasing** sequence (since we're adding positive terms), and they are **bounded above** by 1. By a fundamental theorem of real analysis:

Theorem (SeqConvOfMonotoneBdd): If a sequence $a : \mathbb{N} \to \mathbb{R}$ is monotone and bounded, then it converges.

We have already proven that monotone bounded sequences are Cauchy (`IsCauchyOfMonotoneBdd`), and by the completeness of the real numbers, Cauchy sequences converge (`Conv_of_IsCauchy`).

5.4 The Complete Argument

Let $a_n = \frac{1}{(n+2)^2}$ and let $S_n = \sum_{k=0}^{n-1} a_k$ denote the partial sums.

Step 1: The sequence S_n is monotone increasing.

For any n, we have:

$$S_{n+1} = S_n + a_n$$

Since $a_n = \frac{1}{(n+2)^2} > 0$, we have $S_{n+1} > S_n$.

Step 2: The sequence S_n is bounded above.

Define $b_n = \frac{1}{(n+1)(n+2)}$. We have shown that $a_n \leq b_n$ for all n. By the Series Order Theorem:

$$S_n = \sum_{k=0}^{n-1} a_k \leq \sum_{k=0}^{n-1} b_k = 1 - \frac{1}{n+1} < 1$$

Therefore, S_n is bounded above by 1.

Step 3: Apply the Monotone Bounded Convergence Theorem.

Since S_n is monotone and bounded, it converges. Therefore, the Basel series converges.

5.5 The Formal Proof

```
theorem SeqConvOfMonotoneBdd (a : ℕ → ℝ) (M : ℝ)
    (hM : ∀ n, a n ≤ M) (ha : Monotone a) : SeqConv a :=
        by
  have := IsCauchyOfMonotoneBdd a M hM ha
  exact Conv_of_IsCauchy this
```

```
Statement (a : ℕ → ℝ)
    (ha : ∀ n, a n = 1 / ((n + 2) ^ 2)) : SeriesConv a
      := by
  apply SeqConvOfMonotoneBdd (Series a) 1
  -- Prove boundedness
  let b : ℕ → ℝ := fun n ↦ 1 / ((n + 1) * (n + 2))
  have hb : ∀ n, b n = 1 / ((n + 1) * (n + 2)) := by
    intro n; rfl
  have hab : ∀ n, a n ≤ b n := by
    intro n
    rewrite [ha n, hb n]
    field_simp
    bound  -- (n+2)^2 ≥ (n+1)(n+2)
  intro n
  have bLeib := LeibnizSeries' b hb n
  have habBnd := SeriesOrderThm a b hab n
  change Series b n = 1 - 1 / (n + 1) at bLeib
  have h1 : (1 : ℝ) - 1 / (n + 1) ≤ 1 := by
    field_simp; bound
  linarith [habBnd, h1, bLeib]
  -- Prove monotonicity
  apply Monotone_of_succ
  intro n
  change ∑ k ∈ range n, a k ≤ ∑ k ∈ range (n + 1), a k
  rewrite [show ∑ k ∈ range (n + 1), a k =
   ∑ k ∈ range n, a k + a n by apply sum_range_succ]
  rewrite [ha n]
  have han : (0 : ℝ) ≤ 1 / ((n + 2) ^ 2) := by bound
  linarith [han]
```

5.6 What We Haven't Shown

Notice that we have proven the Basel series converges, but we have *not* computed its exact value. That requires much more sophisticated techniques, which Euler developed using his revolutionary work connecting infinite series to trigonometric functions via the sine function's infinite product representation.

The fact that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ remains one of the most beautiful results in mathematics, connecting the discrete world of integers to the transcendental constant π.

5.7 Further Generalizations

Euler went on to evaluate $\sum_{k=1}^{\infty} \frac{1}{k^{2n}}$ for all positive integers n, showing each equals a rational multiple of π^{2n}. These values are now known as special values of the Riemann zeta function $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$.

The question of whether $\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}$ is a rational multiple of π^3 (or any simple expression involving π) remains open, though we know it's irrational (Apéry's theorem, 1978).

6 Historical Digression: Cesàro Averages and the Quest for the Right Definition of Convergence

Before the modern ε-δ definition of convergence was fully established in the 19th century, mathematicians grappled with sequences and series that seemed to "want" to converge but didn't fit any rigorous framework. This led to fascinating debates about what convergence should really mean.

6.1 Cesàro Averages

One alternative notion of convergence was proposed by Ernesto Cesàro. Instead of looking at the sequence (a_n) itself, we look at the **averages** of its initial terms.

Definition (Cesàro Average): Given a sequence $a : \mathbb{N} \to \mathbb{R}$, its **Cesàro average** (or Cesàro mean) is the sequence $\sigma : \mathbb{N} \to \mathbb{R}$ defined by:

$$\sigma_n = \frac{1}{n} \sum_{k=0}^{n-1} a_k = \frac{a_0 + a_1 + \cdots + a_{n-1}}{n}$$

We say that a sequence is **Cesàro convergent** to L if $\lim_{n\to\infty} \sigma_n = L$.

6.2 Cesàro's Theorem

The key relationship between ordinary convergence and Cesàro convergence is given by the following fundamental result:

Theorem (Cesàro): If a sequence (a_n) converges to L in the usual sense, then its Cesàro averages also converge to L.

That is: ordinary convergence implies Cesàro convergence to the same limit.

6.3 Sketch of Proof

Suppose $a_n \to L$. We want to show that $\sigma_n \to L$, where:

$$\sigma_n = \frac{1}{n} \sum_{k=0}^{n-1} a_k$$

The key idea is to split the sum into two parts: an initial segment (where a_k might not be close to L) and a tail segment (where a_k is close to L).

Let $\varepsilon > 0$ be given. Since $a_n \to L$, there exists N_0 such that for all $k \geq N_0$:

$$|a_k - L| < \frac{\varepsilon}{2}$$

Now, for $n > N_0$, we can write:

$$\begin{aligned}
\sigma_n - L &= \frac{1}{n}\sum_{k=0}^{n-1} a_k - L \\
&= \frac{1}{n}\sum_{k=0}^{n-1}(a_k - L) \\
&= \frac{1}{n}\sum_{k=0}^{N_0-1}(a_k - L) + \frac{1}{n}\sum_{k=N_0}^{n-1}(a_k - L)
\end{aligned}$$

For the first sum, note that it has only N_0 terms, each of which is fixed. So:

$$\left|\frac{1}{n}\sum_{k=0}^{N_0-1}(a_k - L)\right| \leq \frac{1}{n}\sum_{k=0}^{N_0-1}|a_k - L| \leq \frac{C}{n}$$

for some constant C depending on the first N_0 terms. For n large enough (say $n \geq N_1$ where $C/N_1 < \varepsilon/2$), this is less than $\varepsilon/2$.

For the second sum, each term satisfies $|a_k - L| < \varepsilon/2$, so:

$$\left|\frac{1}{n}\sum_{k=N_0}^{n-1}(a_k - L)\right| \leq \frac{1}{n}\sum_{k=N_0}^{n-1}|a_k - L| < \frac{1}{n} \cdot n \cdot \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$$

Combining these, for all $n \geq \max(N_0, N_1)$:

$$|\sigma_n - L| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This completes the proof. □

6.4 The Converse is False!

Crucially, the converse of Cesàro's theorem is **false**: a sequence can be Cesàro convergent without being convergent in the usual sense. This is where the historical confusion arose.

6.5 The Grandi Series Paradox

Consider the alternating sequence $a_n = (-1)^n = (1, -1, 1, -1, 1, -1, \ldots)$.

Does this sequence converge? In the modern sense, no! The sequence oscillates between 1 and -1, never approaching any single value. For any proposed limit L and any $\varepsilon < 1$, we cannot find an N such that all terms after N are within ε of L.

But what about its Cesàro average? Let's compute:

$$\begin{aligned}
\sigma_1 &= 1 \\
\sigma_2 &= \frac{1 + (-1)}{2} = 0 \\
\sigma_3 &= \frac{1 + (-1) + 1}{3} = \frac{1}{3} \\
\sigma_4 &= \frac{1 + (-1) + 1 + (-1)}{4} = 0 \\
\sigma_5 &= \frac{1 + (-1) + 1 + (-1) + 1}{5} = \frac{1}{5}
\end{aligned}$$

In general, for even $n = 2m$, we have $\sigma_n = 0$. For odd $n = 2m + 1$, we have $\sigma_n = \frac{1}{2m+1}$.

Therefore: $\lim_{n\to\infty} \sigma_n = 0$.

The Cesàro average converges to 0!

6.6 The Associated Series: Even More Confusion

Now consider the **Grandi series**, the infinite series associated with this sequence:

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

The partial sums are:

$$S_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

This series does **not** converge in the usual sense – the partial sums oscillate between 0 and 1.

But what about the Cesàro average of the partial sums? The Cesàro average is:

$$\frac{1}{n}\sum_{k=1}^{n} S_k$$

For large even $n = 2m$:

$$\frac{1}{2m}(S_1 + S_2 + \cdots + S_{2m}) = \frac{1}{2m}(1 + 0 + 1 + 0 + \cdots + 1 + 0) = \frac{m}{2m} = \frac{1}{2}$$

Therefore, the Cesàro sum of the Grandi series is $\boxed{\frac{1}{2}}$.

6.7 The Historical Confusion

Before the modern definition of convergence was settled, mathematicians debated questions like:

Does the sequence $(-1)^n$ converge? If so, to what value?

Possible answers:

- **Answer 1:** No, it doesn't converge at all – it oscillates forever.
- **Answer 2:** Yes, it converges to 0 in the Cesàro sense.
- **Answer 3:** The associated series has Cesàro sum $\frac{1}{2}$, so perhaps the "average value" is $\frac{1}{2}$?

And more confusingly:

What is the value of $1 - 1 + 1 - 1 + 1 - 1 + \cdots$?

Different methods gave different answers! Guido Grandi (1703) argued it should equal $\frac{1}{2}$ by writing:

$$(1 - 1) + (1 - 1) + (1 - 1) + \cdots = 0 + 0 + 0 + \cdots = 0$$

But also:

$$1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots = 1 + 0 + 0 + 0 + \cdots = 1$$

Therefore, by "averaging," the sum is $\frac{1}{2}$!

Leibniz endorsed this reasoning, arguing that $\frac{1}{2}$ represented a kind of "middle" or "equilibrium" value. Euler also accepted that such series could have values, though he was aware of the paradoxes.

6.8 The Resolution: Cauchy and Weierstrass

The confusion was finally resolved in the 19th century by Augustin-Louis Cauchy (1821) and later, more rigorously, by Karl Weierstrass (1850s-1860s). They established the modern ε-N definition of convergence:

> A sequence (a_n) converges to L if: for every $\varepsilon > 0$, there exists N such that for all $n \geq N$, $|a_n - L| < \varepsilon$.

Under this definition:

- The sequence $(-1)^n$ does **not** converge.
- The series $\sum(-1)^k$ does **not** converge.
- All paradoxes disappear: we simply declare these sequences/series divergent.

Cesàro averages remain interesting as a **summability method** – a way of assigning "generalized sums" to certain divergent series – but they are clearly distinguished from ordinary convergence.

6.9 Why This Definition Won

The ε-N definition became standard for several reasons:

1. Consistency: It eliminates contradictions. We no longer have to decide between competing "values" for divergent series.

2. Algebraic properties: If $a_n \to L$ and $b_n \to M$ (in the ε-N sense), then:

- $a_n + b_n \to L + M$
- $a_n \cdot b_n \to L \cdot M$
- If $M \neq 0$, then $a_n/b_n \to L/M$ (eventually)

These properties hold cleanly for ε-N convergence but require additional hypotheses for Cesàro convergence.

3. Completeness: The ε-N definition plays perfectly with the completeness of $\mathbb{R}$. A sequence converges if and only if it is Cauchy, giving us powerful tools for proving convergence without knowing the limit in advance.

4. Analysis works: Limits of continuous functions, derivatives, integrals – all the machinery of analysis works smoothly with this definition.

6.10 Modern Perspective

Today, we view the situation clearly:

- **Convergence** (in the ε-N sense) is the fundamental notion.
- **Cesàro convergence** is a weaker notion: it's a type of *summability*.
- If a sequence converges, it Cesàro-converges to the same limit.
- The converse is false: Cesàro convergence doesn't imply convergence.

Cesàro summability is part of a broader theory of summability methods (Abel summation, Borel summation, etc.), used in analytic number theory, Fourier analysis, and other advanced topics. These methods can assign "generalized values" to divergent series in useful ways – for instance, the Ramanujan summation $1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$ appears in string theory!

But for basic real analysis, the standard ε-N definition reigns supreme, and we're grateful to Cauchy and Weierstrass for clarifying the confusion of earlier eras.

6.11 A Final Thought: The Importance of Definitions

This historical episode teaches us a profound lesson: **mathematics progresses not just by proving theorems, but by finding the right definitions**.

Before Cauchy, mathematicians had powerful intuitions about limits and continuity, but they lacked precise definitions. This led to errors, paradoxes, and endless debates. Cauchy's achievement wasn't discovering new facts about limits – it was finding the right *definition* that made all the facts clear.

As you continue in mathematics, you'll encounter this pattern repeatedly. Often the hardest part of solving a problem is figuring out what question you're really asking. Once the definitions are right, the theorems often follow naturally.

Theorem: If $\underbrace{\sum a_n \overset{(ha)}{=} L \text{ converges}}$, then $\underbrace{a_n \to 0}$.

Sketches:

Approach 1: If not, then

$\exists \varepsilon > 0, \forall N, \exists n > N$ s.t. $|a_n| \geq \varepsilon$. Then $\left|\sum_{k=0}^{n-1} a_k - L\right| < \varepsilon/2$

but $\left|\sum_{k=0}^{n} a_k - L\right| < \varepsilon/2$.

Sketch of direct pf:

$(a_n) \Rightarrow$ series a is Cauchy:

i.e. $\forall \varepsilon > 0\ \exists N\ \forall n \geq N\ \forall m \geq n$,

know: $\underbrace{\left| \sum^{m-1} a_k - \sum^{n-1} a_k \right| < \varepsilon}_{|a_n|}$.

apply to $m := n+1$

---

pf: intro ε $h\varepsilon$.

choose N hN using IsCauchy of lim ha

use N $\qquad \boxed{\text{Goal: } |a_n - 0| < \varepsilon}$ ε $h\varepsilon$,

intro n hn; $\quad \boxed{hN : \left| \text{series } a\,(n+1) - \text{series } n \right| < \varepsilon}$

specialize hN n hn $(n+1)$ (by bound)

change $\left| \sum k \in \text{range}(n+1), a\,k - \sum k \in \text{range } n, a\,k \right| < \varepsilon$ at hN

rewrite [show ∑ k ∈ range (n+1), a k =
∑ k ∈ range n, a k + a n, by apply sum_range_succ] at hN

rewrite [show ∑ n, a k + a n = ∑ n, a k = a n by ring_nf] at hN

rewrite [show a n - 0 = a n by ring]
apply hN

$$a_n := \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}.$$

Problem: $\sum a_n = ?$.

a_n: $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \dots$

$\sum_{k \in \mathrm{range}\, n} = \sum_{0}^{n-1}$

$b_n = \sum^n a_k : 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots$ ✓

Claim: $b_n = 1 - \frac{1}{n+1}$

induction $(n=0)$. $b_0 = 0 = 1 - \frac{1}{0+1} = 0$,

Assume h_n: $b_n = 1 - \frac{1}{n+1}$

Goal: $b_{n+1} = 1 - \frac{1}{n+1+1}$.

$b_{n+1} = b_n + a_n = \left(1 - \cancel{\frac{1}{n+1}}\right) + \cancel{\frac{1}{n+1}} - \frac{1}{n+2}$. ✓

$\sum : \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \frac{1}{49}, \dots$ $\sim$ 300 Archimedes

(400 Madhava)

$\boxed{1660 \text{ Newton}}$

1670 Leibniz

↳ J.J. Bernoulli!

(1760 ↳ Euler

$a_n := \frac{1}{(n+2)^2}$.

$\frac{1}{4} + \frac{1}{9} = \frac{13}{36}$ 0,3.

$\frac{13}{36} + \frac{1}{16} = \frac{161}{144}$. 0,32.

Basel Problem! What is $\sum = \frac{\pi^2}{6} - 1$

Goal: Show $\sum \frac{1}{(n+2)^2}$ converges.

idea: bound by $\frac{1}{(n+2)(n+2)} \le \frac{1}{(n+1)(n+2)}$

$\sum \le \infty$

sketch: $\sum \frac{1}{(n+2)^2} \le \sum \frac{1}{(n+1)(n+2)} = 1.$

(1) Bounded (seq of partial sums)

(2) Monotone.

Why if $a_n \to L$ does

Cesàro avg $b_n := \frac{a_0 + \dots + a_{n-1}}{n}$.

also conv to L?

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} (a_k - L) \right| \overset{?}{<} \varepsilon$$

$\exists N_1$ s.t. $\forall n \geq N_1$, $|a_n - L| < \varepsilon/2$.

$\forall n, |a_n| \leq M$. , $\dfrac{N_1 \cdot (M + |L|)}{n} < \dfrac{\varepsilon}{2}$

$$n \geq N_2 > \frac{2N_1 \cdot (M + |L|)}{\varepsilon}.$$