An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 11: The Real Numbers I

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1 Introduction: Cauchy Sequences and the Real Numbers

SOCRATES: So far we've learned that *if* a sequence converges, then it's bounded, and moreover that any subsequence of it also converges to the same limit.

SIMPLICIO: Yeah, so what?

SOCRATES: And we saw that there can be sequences which do not themselves converge – for example, $(-1)^n$ – but which are bounded and have subsequences that do converge. The even-indexed terms, in this example, are all equal 1.

SIMPLICIO: What are you getting at?

SOCRATES: Well, what's a question that a mathematician might naturally ask given that information?

SIMPLICIO: You mean whether that always happens?

SOCRATES: Yes, something like that. Can you elaborate?

SIMPLICIO: Okay, I'll play along. You're trying to get me to formulate some kind of converse. If a sequence is bounded, then... it converges? No, that can't be right – a bounded sequence can bounce around without converging, like $(-1)^n$ itself.

Ah, but maybe there's always *some* subsequence that converges? Hmm, but that can't be right either, since the sequence $a_n = n$ has no convergent subsequence – it just escapes to infinity.

Oh! But wait, that sequence isn't bounded. Are you saying that if all I know about a sequence is that it's bounded, then there's always *some* subsequence that converges?

SOCRATES: Yes, precisely! This important fact is called the "Bolzano-Weierstrauss theorem". But here's where it gets **really** subtle. Think about the sequence of fractions: $a(0) = 1/1$, $a(1) = 14/10$, $a(2) = 141/100$, $a(3) = 1414/1000$, ... getting closer and closer to $1.4142\cdots = \sqrt{2}$. The sequence is bounded (by 2, to be crude), and even increasing, but its limit is not a rational number! So the Bolzano-Weierstrauss theorem is not true for the rationals. As I warned you long ago, we'll have to eventually face the fact that we don't even know what the real numbers *are*. I think that time is now.

SIMPLICIO: Fine, I'm ready; tell me what they are.

SOCRATES: Unfortunately, it's rather complicated, and it'll take us some time to arrive at the answer, and to see why it *is* the answer. Let's take a step back. What would you *like* to be able to say about the real numbers?

SIMPLICIO: Well, I guess I'd like to say something like: they're the limits of their decimal expansions. So they're limits of rational sequences. Like, $\sqrt{2}$ is the limit of that sequence you just mentioned: $1, 1.4, 1.41, 1.414, \ldots$

SOCRATES: Good! So you want to define a real number as "the limit of a sequence of rationals." But remind me, what does it mean for a sequence to have a limit?

SIMPLICIO: It means that for all $\varepsilon > 0$, there exists an N, yadda yadda. The terms get arbitrarily close to some number L.

SOCRATES: And what is this mysterious number L? What *type* of number is it?

SIMPLICIO: It's... a real number. Oh no.

SOCRATES: Exactly! We have a circular definition. We're trying to define the real numbers as limits of rational sequences, but the very notion of "limit" that we've been using presupposes that we already know what the real numbers are!

SIMPLICIO: So we're stuck? We can't define the real numbers?

SOCRATES: Sure seems like it! But this is where Cauchy had a **brilliant** insight. He realized the same thing you did: he can't use real numbers to define limits. He wants to say: "a_n gets closer and closer to L" but without reference to L itself. He needs to find *something else* that he can say a_n gets close to.

SIMPLICIO: But he *has* nothing else.

SOCRATES: Exactly!! So...?

SIMPLICIO: So if all he has is the sequence a_n, and he has to compare it to something, and he has nothing else... Oh!!! He has to compare it to **itself**!?! But how?

SOCRATES: Wow, you got it! Yes, exactly, How?

SIMPLICIO: Well of course it's pointless to ask if $|a_n - a_n| < \varepsilon$. But... you could ask for $|a_n - a_m| < \varepsilon$, once n and m are *both* large enough?

SOCRATES: Ha, you did it! Yes, exactly, if a_n and a_m are both within ε of **each other**, that should be a substitute for convergence.

SIMPLICIO: That's so clever! So instead of saying "the sequence converges to L," we say "the terms of the sequence get arbitrarily close to each other"?

SOCRATES: Precisely. Can you make this formal, using ε's and N's?

SIMPLICIO: I think so. I guess we should say that a sequence a_n has a limit if: for every $\varepsilon > 0$, there exists an N such that for all $m, n \geq N$, we have $|a_m - a_n| < \varepsilon$.

SOCRATES: Beautiful! But since we already have a different meaning for the notion of "has a limit", we'll call this property "Cauchy". So we say that **a sequence is Cauchy** if, as you said:

$$\forall \varepsilon > 0, \exists N, \forall m \geq N, \forall n \geq N, |a_m - a_n| < \varepsilon$$

This is one of the most important definitions in **all of mathematics**. It appears not only here in real analysis, but also in higher arithmetic when building the p-adic numbers, in functional analysis when studying Banach spaces and Hilbert spaces, and in topology and geometry when "completing" metric spaces. Anywhere mathematicians want to talk about "convergence" but without knowing *a priori* where things converge *to*, they reach for a version of Cauchy's definition.

But before we return to the real numbers, let's first get more familiar with this definition and what it can do.

SIMPLICIO: I like it; let's go!

2 Big Boss: Limits are Cauchy

One of the fundamental relationships in analysis is that convergence implies the Cauchy property. This theorem establishes that any sequence with a limit satisfies Cauchy's self-referential criterion for convergence.

2.1 The Mathematical Setup

Definition (IsCauchy): A sequence $a : \mathbb{N} \to \mathbb{R}$ is said to be **Cauchy** if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have $|a_m - a_n| < \varepsilon$.

Theorem: If a sequence $a : \mathbb{N} \to \mathbb{R}$ converges, then it is Cauchy.

2.2 New Tools

abs_sub_comm: For any real numbers x and y, we have $|x - y| = |y - x|$.

This symmetry property is crucial for manipulating absolute value expressions involving differences.

2.3 Strategic Approach

The proof strategy involves:

- Using the convergence hypothesis to get close to the limit L
- Applying the triangle inequality to relate $|a_m - a_n|$ to distances from the limit
- Choosing $\varepsilon/2$ when applying the limit definition to ensure the sum stays below ε

The key insight is that if both a_m and a_n are within $\varepsilon/2$ of L, then they must be within ε of each other.

2.4 Lean Solution

```
Statement IsCauchyOfLim (a : ℕ → ℝ) (ha : SeqConv a)
    : IsCauchy a := by
choose L hL using ha
```

```
intro ε hε
choose N hN using hL (ε / 2) (by bound)
use N
intro n hn m hm
have hn' : |a n - L| < ε / 2 := by apply hN n hn
have hm' : |a m - L| < ε / 2 := by apply hN m hm
rewrite [(by ring_nf : |a n - a m| = |(a n - L) + (L - a
    m)|)]
have f1 : |(a n - L) + (L - a m)| ≤ |a n - L| + |L - a m
   | := by apply abs_add
have f2 : |L - a m| = |a m - L| := by apply abs_sub_comm
linarith [f1, f2, hn', hm']
```

2.5 Natural Language Proof

Proof: Assume a converges. Then there exists $L \in \mathbb{R}$ such that $a_n \to L$.

Given $\varepsilon > 0$, since $a_n \to L$, there exists N such that for all $n \geq N$:

$$|a_n - L| < \frac{\varepsilon}{2}$$

Now for any $m, n \geq N$, we have:

$$\begin{aligned}
|a_m - a_n| &= |(a_m - L) + (L - a_n)| \\
&\leq |a_m - L| + |L - a_n| \\
&= |a_m - L| + |a_n - L| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Therefore, a is Cauchy. **QED**

3 Level 2: Sums of Cauchy Sequences

Just as sums of convergent sequences converge, sums of Cauchy sequences are Cauchy. This result is important because it shows that the Cauchy property is preserved under arithmetic operations.

3.1 The Mathematical Setup

Theorem: If sequences a and b are Cauchy, then their sum $a + b$ is also Cauchy.

3.2 Strategic Approach

The proof follows a similar pattern to the sum of limits theorem:

- Apply the Cauchy property to both a and b with $\varepsilon/2$
- Take N to be the maximum of the two individual N values
- Use the triangle inequality to combine the estimates

3.3 Lean Solution

```
Statement IsCauchyOfSum (a b : ℕ → ℝ) (ha : IsCauchy a)
   (hb : IsCauchy b)
    : IsCauchy (a + b) := by
intro ε hε
choose N1 hN1 using ha (ε / 2) (by bound)
choose N2 hN2 using hb (ε / 2) (by bound)
use N1 + N2
intro m hm n hn
specialize hN1 m (by bound) n (by bound)
specialize hN2 m (by bound) n (by bound)
change |(a m + b m) - (a n + b n)| < ε
rewrite [(by ring_nf : |(a m + b m) - (a n + b n)| = |(a
    m - a n) + (b m - b n)|)]
have f1 : |a m - a n + (b m - b n)| ≤ |a m - a n| + |(b
   m - b n)| := by apply abs_add
linarith [f1, hN1, hN2]
```

3.4 Natural Language Proof

Proof: Assume a and b are Cauchy sequences.

Given $\varepsilon > 0$, since a is Cauchy, there exists N_1 such that for all $m, n \geq N_1$:

$$|a_m - a_n| < \frac{\varepsilon}{2}$$

Similarly, since b is Cauchy, there exists N_2 such that for all $m, n \geq N_2$:

$$|b_m - b_n| < \frac{\varepsilon}{2}$$

Let $N = N_1 + N_2$. For any $m, n \geq N$, we have:

$$\begin{aligned}
|(a+b)_m - (a+b)_n| &= |(a_m + b_m) - (a_n + b_n)| \\
&= |(a_m - a_n) + (b_m - b_n)| \\
&\leq |a_m - a_n| + |b_m - b_n| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Therefore, $a + b$ is Cauchy. **QED**

4 Level 3: Cauchy Implies Bounded

One of the key properties of Cauchy sequences is that they are bounded. This parallels the result that convergent sequences are bounded, but now we prove it directly from the Cauchy property without reference to a limit.

4.1 The Mathematical Setup

Theorem: If a sequence $a : \mathbb{N} \to \mathbb{R}$ is Cauchy, then it is bounded.

4.2 Strategic Approach

The proof strategy involves:

- Using the Cauchy property with $\varepsilon = 1$ to ensure that eventually all terms stay within distance 1 of some fixed term
- Handling the finitely many initial terms separately by taking their maximum
- Combining these two bounds to get an overall bound for the entire sequence

The key insight is that a Cauchy sequence is "eventually clustered," and we only need to account for finitely many terms before this clustering begins.

4.3 Lean Solution

```
Statement IsBddOfCauchy (a : ℕ → ℝ) (ha : IsCauchy a)
    : SeqBdd a := by
choose N hN using ha 1 (by bound)
use |a N| + 1 + ∑ k ∈ range N, |a k|
have aNnonneg : 0 ≤ |a N| := by bound
have sumNonneg : 0 ≤ ∑ k ∈ range N, |a k| := by apply
   sum_nonneg (by bound)
have f1 : ∀ n < N, |a n| ≤ ∑ k ∈ range N, |a k| := by
   apply TermLeSum a N
split_ands
linarith [aNnonneg, sumNonneg]
intro n
```

```
specialize hN N (by bound) n
by_cases hn : n < N
specialize f1 n hn
linarith [f1, aNnonneg]
specialize hN (by bound)
have f2 : |a n| = |(a n - a N) + a N| := by ring_nf
have f3 : |(a n - a N) + a N| ≤ |a n - a N| + |a N| :=
   by apply abs_add
rewrite [(by apply abs_sub_comm : |a n - a N| = |a N - a
    n|)] at f3
linarith [f2, f3, hN, sumNonneg]
```

4.4 Natural Language Proof

Proof: Assume a is Cauchy.

Since a is Cauchy, taking $\varepsilon = 1$, there exists N such that for all $m, n \geq N$:

$$|a_m - a_n| < 1$$

In particular, for all $n \geq N$:

$$|a_n - a_N| < 1$$

By the triangle inequality:

$$|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| < 1 + |a_N|$$

For the finitely many terms with $n < N$, let:

$$M_0 = \max\{|a_0|, |a_1|, \ldots, |a_{N-1}|\}$$

Then for all $n \in \mathbb{N}$:

$$|a_n| \leq \max\{M_0, 1 + |a_N|\}$$

Therefore, a is bounded. **QED**

4.5 Significance

This theorem is crucial for the construction of the real numbers. It shows that Cauchy sequences share a fundamental property with convergent sequences, even though we haven't yet established that Cauchy sequences converge (in the rationals, they don't always!). This boundedness will be essential when we eventually prove the Bolzano-Weierstrauss theorem and establish that the real numbers are "complete"—that is, every Cauchy sequence of real numbers converges to a real number.

Last time:

Saw: if $a : \mathbb{N} \to \mathbb{R}$ converges,

then (1) a is bounded, i.e.,

IsBddofConv $\exists M > 0, \forall n, |a_n| \le M$.

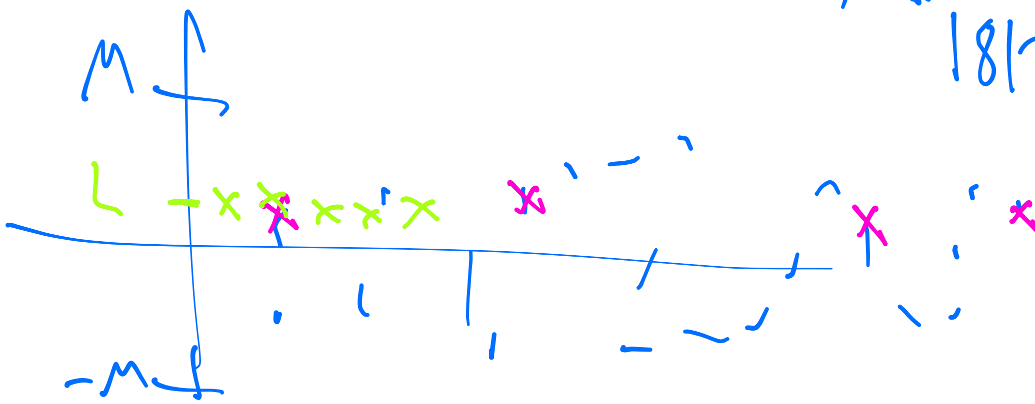
(2) $\forall \sigma : \mathbb{N} \to \mathbb{N}$ subseq ($\forall i<j, \sigma_i < \sigma_j$),

SubseqConv. $a \circ \sigma$ also converges & to the same thing.

Obs: seq $a_n = (-1)^n$ does not converge, but is bounded & it has a subseq ($\sigma(n) = 2n$) which does converge.

Natural Q: is it true that all bounded sequences have subsequences

that converge? (Yes, Bolzano-Weierstrass)
"1817" 1860s



B-W is NOT TRUE for $\mathbb{Q}$, a_n: $\frac{1}{1}, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \ldots$

$1, 1.4, 1.41, 1.414, 1.4142, \ldots \leq 2$,

this is an increasing, bounded sequence in $\mathbb{Q}$, which does not have a limit in $\mathbb{Q}$ (limit is $\sqrt{2} \notin \mathbb{Q}$).

<u>Idea:</u> $\mathbb{R} \overset{???}{=}$ <u>limits of $\mathbb{Q}$</u>.

Use $\mathbb{R}$ in def, no good

Cauchy: instead of $a_n \to L$,

get $a_n \approx a_m$. $|a_n - \cancel{a_n}| < \varepsilon$.

$|a_n - a_m| < \varepsilon$, once n, m both large

<u>Def</u>: A sequence $a: \mathbb{N} \to X$ $(X = \mathbb{Q}, X = \mathbb{R})$, is <u>Cauchy</u> if

$\forall \varepsilon > 0, \exists N, \forall m \geq N, \forall n \geq N,$

$$|a_m - a_n| < \varepsilon.$$

<u>Q</u>: Can we compare n & $n+1$?

For our essential purposes: $\left|\frac{-7}{15}\right| = \frac{7}{15}$.

Seq of rationals $a: \mathbb{N} \to \mathbb{Q}$ is

Cauchy iff $\forall (\varepsilon : \mathbb{Q}), \varepsilon > 0 \to \exists (N : \mathbb{N}), \forall m, n \geq N, |a_n - a_m| < \varepsilon$,

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Thm: Given $a : \mathbb{N} \to \mathbb{R}$ & ha : SeqConv a, Goal: IsCauchy a,

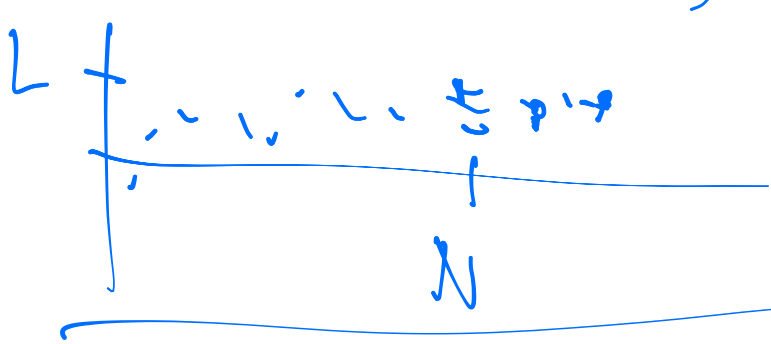
Change $\forall \varepsilon > 0, \exists N, \forall m \geq N, \forall n \geq N,$
$|a_m - a_n| < \varepsilon$.

intro ε $h\varepsilon$, ("Let ε be given & assume $\varepsilon > 0$"),

Choose L hL using ha,

hL : SeqConv a L

$hL : \forall \varepsilon > 0, \exists N, \forall n \geq N, |a_n - L| < \varepsilon$.



Idea get a_n & a_m within $\varepsilon/2$ of L

Choose N hN using hL $(\varepsilon/2)$ (by bound) use N | Goal: $\forall m \geq N, \forall n \geq N, \ldots$ | $hN: \forall n \geq N, |a_n - L| < \frac{\varepsilon}{2}$,

intro m hm n hn | Goal: $|a_m - a_n| < \varepsilon$,

have hm' : $|a_m - L| < \varepsilon/2$:= by apply hN m hm
have hn' : $|a_n - L| < \varepsilon/2$:= by apply hN n hn

rewrite [(by ring_nf : $a_m - a_n = (a_m - L) + (L - a_n)$)]

have f1 : $|\_ + \_|$ | Goal: $|(a_m - L) + (L - a_n)| < \varepsilon$,

$\leq |a_m - L| + |L - a_n|$:= by apply abs_add,

rewrite [(by apply abs_sub_comm : $|L - a_n| = |a_n - L|$)] at f1,

linarith [f1, hm', hn'].

Theorem: If a & b are Cauchy, so is a+b (: $\mathbb{N} \to \mathbb{R}$).

intro ε hε.

Choose N1 hN1 using ha (ε/2) (by, done)

Choose N2 hN2 using hb (ε/2) (by, done)

use N_1+N_2.

intro m hm n hn. Goal: $|(a+b)_m - (a+b)_n| < ε$

change $|(a_m+b_m) - (a_n+b_n)| < ε$,

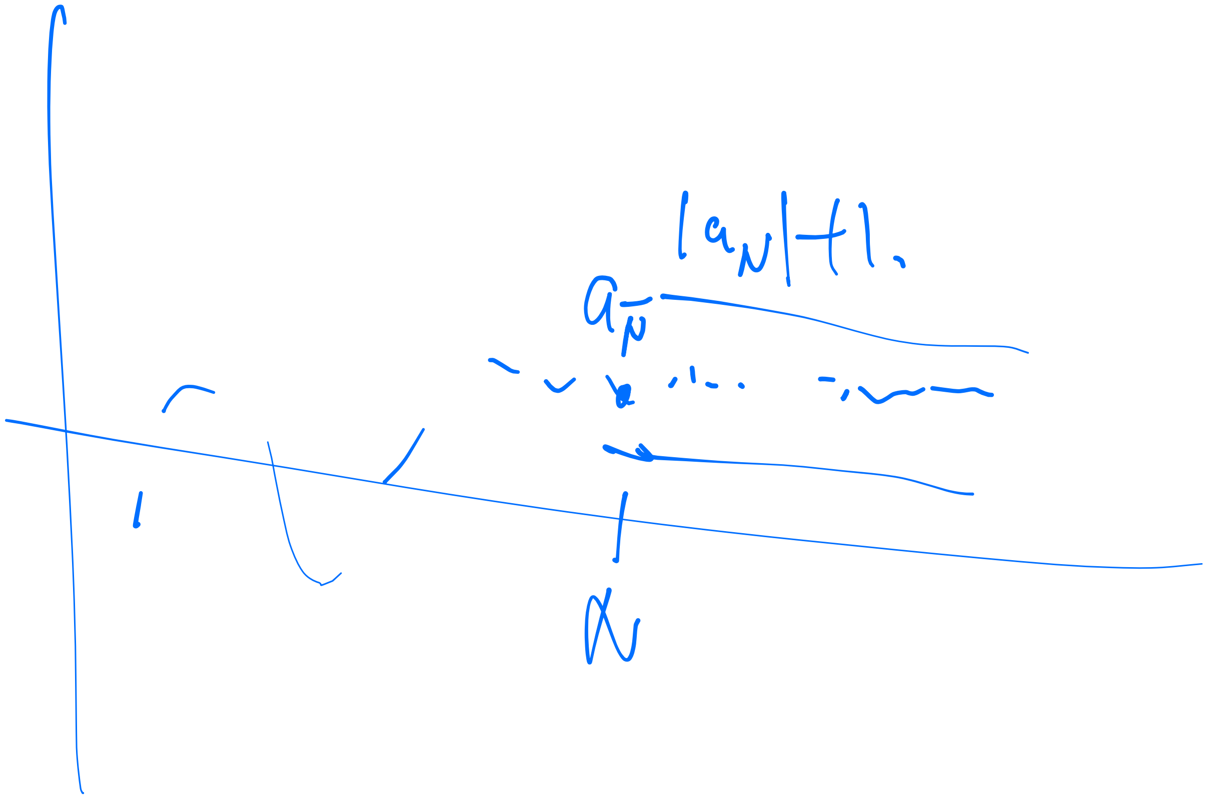
rewrite [(by ring : $a_m+b_m-(a_n+b_n)$ $= (a_m-a_n)+(b_m-b_n)$)]

have f1 : $|(a_m-a_n)+(b_m-b_n)| \le |a_m-a_n|+|b_m-b_n|$:= by ---

have ha' : $|a_m-a_n| < ε/2$:= by apply hN1 m (by, done) n (by, done)

have hb' : — — —

linarith [f1, ha', hb'].



|a_N|+1
a_N
N