An Introduction to Formal Real Analysis, Rutgers University, Fall 2025, Math 311H

Lecture 10: Algebraic Limit Theorem and Order Properties

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1 Product of Sequences

Having established the Algebraic Limit Theorem for sums and scalar multiples, we now tackle the product of convergent sequences. This result, combined with our earlier work, completes the algebraic structure of limits: we can add, multiply by constants, multiply sequences together, and even divide (provided the limit is nonzero).

The proof strategy mirrors the product rule from calculus, where $(fg)' = f'g + g'f$. We artificially introduce terms that allow us to separate the contributions from each sequence.

1.1 The Mathematical Statement

Theorem (ProdLimNeNe): If $a, b : \mathbb{N} \to \mathbb{R}$ are sequences with $a(n) \to L$, $b(n) \to M$, where $L \neq 0$ and $M \neq 0$, and $c(n) = a(n) \cdot b(n)$ for all n, then:

$$c(n) \to L \cdot M$$

1.2 Strategic Approach

The key insight is to write:

$$a(n) \cdot b(n) - L \cdot M = (a(n) - L) \cdot b(n) + L \cdot (b(n) - M)$$

This decomposition separates the error into two parts:

- $(a(n) - L) \cdot b(n)$: the error in a times the value of b
- $L \cdot (b(n) - M)$: the limit L times the error in b

To bound the first term, we use that b is convergent and hence bounded (by some $K > 0$). Then $|a(n) - L|$ can be made less than $\varepsilon/(2K)$, giving us control over the first term.

For the second term, we directly use that $b(n) \to M$, making $|b(n) - M|$ less than $\varepsilon/(2|L|)$.

1.3 Lean Solution

```
Statement ProdLimNeNe (a b c : ℕ → ℝ) (L M : ℝ) (hL : L
  ≠ 0) (hM : M ≠ 0) (ha : SeqLim a L)
    (hb : SeqLim b M) (hc : ∀ n, c n = a n * b n):
    SeqLim c (L * M) := by
intro ε hε
choose K hK using BddOfConvNonzero b M hb hM
have ε1 : 0 < ε / (2 * K) := by bound
have absL : 0 < |L| := by apply abs_pos_of_nonzero hL
have ε2 : 0 < ε / (2 * |L|) := by bound
specialize ha (ε / (2 * K)) ε1
specialize hb (ε / (2 * |L|)) ε2
choose N1 hN1 using ha
choose N2 hN2 using hb
use N1 + N2
intro n hn
have hn1 : N1 ≤ n := by bound
have hn2 : N2 ≤ n := by bound
specialize hN1 n hn1
specialize hN2 n hn2
specialize hc n
rewrite [hc]
have f1 : |a n * b n - L * M| = |(a n - L) * b n + (L *
  (b n -  M))| := by ring_nf
have f2 : |(a n - L) * b n + (L * (b n -  M))| ≤ |(a n -
    L) * b n| + |(L * (b n -  M))| := by apply abs_add
```

```
have f3 : |(a n - L) * b n| = |(a n - L)| * |b n| := by
    apply abs_mul
have bnBnd : |b n| ≤ K := by apply hK.2 n
have f5 : |(a n - L)| * |b n| ≤ ε / (2 * K) * K := by
    bound
have Kpos : 0 < K := by apply hK.1
have f6 : ε / (2 * K) * K = ε / 2 := by field_simp
have f7 : |(L * (b n -  M))| = |L| * |b n -  M| := by
    apply abs_mul
have f8 :  |L| * |b n -  M| < |L| * (ε / (2 * |L|)) :=
    by bound
have f9 : |L| * (ε / (2 * |L|)) = ε / 2 := by field_simp
linarith [f1, f2, f3, f5, f6, f7, f8, f9]
```

1.4 Natural Language Proof

Proof: Let $\varepsilon > 0$ be given. Since $b(n) \to M$ with $M \neq 0$, by `BddOfConvNonzero` , there exists $K > 0$ such that $|b(n)| \leq K$ for all n.

Define:

$$\varepsilon_1 := \frac{\varepsilon}{2K} \quad \text{and} \quad \varepsilon_2 := \frac{\varepsilon}{2|L|}$$

Note that $\varepsilon_1 > 0$ (since $K > 0$) and $\varepsilon_2 > 0$ (since $L \neq 0$ implies $|L| > 0$).

Since $a(n) \to L$, there exists N_1 such that for all $n \geq N_1$:

$$|a(n) - L| < \varepsilon_1 = \frac{\varepsilon}{2K}$$

Since $b(n) \to M$, there exists N_2 such that for all $n \geq N_2$:

$$|b(n) - M| < \varepsilon_2 = \frac{\varepsilon}{2|L|}$$

Let $N := N_1 + N_2$. For any $n \geq N$, we have $n \geq N_1$ and $n \geq N_2$.

Now, using the algebraic identity:

$$a(n) \cdot b(n) - L \cdot M = (a(n) - L) \cdot b(n) + L \cdot (b(n) - M)$$

Taking absolute values and applying the triangle inequality:

$$\begin{aligned}
|c(n) - L \cdot M| &= |a(n) \cdot b(n) - L \cdot M| \\
&= |(a(n) - L) \cdot b(n) + L \cdot (b(n) - M)| \\
&\leq |(a(n) - L) \cdot b(n)| + |L \cdot (b(n) - M)| \\
&= |a(n) - L| \cdot |b(n)| + |L| \cdot |b(n) - M|
\end{aligned}$$

For the first term, since $|b(n)| \leq K$ and $|a(n) - L| < \varepsilon/(2K)$:

$$|a(n) - L| \cdot |b(n)| < \frac{\varepsilon}{2K} \cdot K = \frac{\varepsilon}{2}$$

For the second term, since $|b(n) - M| < \varepsilon/(2|L|)$:

$$|L| \cdot |b(n) - M| < |L| \cdot \frac{\varepsilon}{2|L|} = \frac{\varepsilon}{2}$$

Therefore:

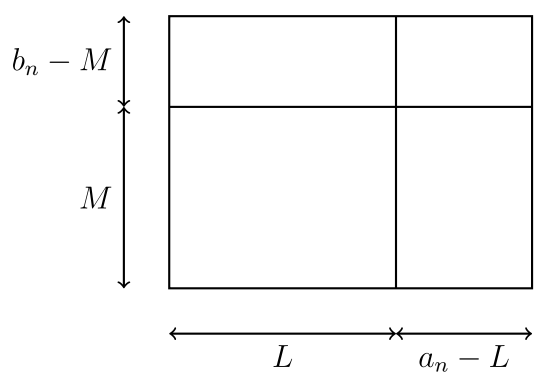
$$|c(n) - L \cdot M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This completes the proof. **QED**

1.5 Connection to Calculus: Why Add and Subtract $b(n) \cdot L$?

The proof technique mirrors the product rule from calculus: $(fg)' = f'g + g'f$. To understand why we add and subtract $b(n) \cdot L$, think geometrically about the area of a rectangle. (After all, multiplication *is* rectangles.)

When both $a(n)$ and $b(n)$ are close to their limits L and M, the product $a(n) \cdot b(n)$ represents the area of a rectangle with dimensions $a(n)$ by $b(n)$, while $L \cdot M$ is the area of the limiting rectangle with dimensions L by M. The difference in areas can be visualized by drawing both rectangles:



The total difference is:

$$a(n) \cdot b(n) - L \cdot M = L \cdot (b(n) - M) + M \cdot (a(n) - L) + (a(n) - L) \cdot (b(n) - M)$$

The key insight is that we can replace M with $b(n)$ in the middle term (adding and subtracting $b(n) \cdot L$) to get:

$$a(n) \cdot b(n) - L \cdot M = (a(n) - L) \cdot b(n) + L \cdot (b(n) - M)$$

Why is this better? Because now each term isolates one source of error:

- $(a(n) - L) \cdot b(n)$: small error in a times the bounded quantity $b(n)$
- $L \cdot (b(n) - M)$: the fixed constant L times small error in b

This decomposition allows us to control each piece separately. The original $(a(n) - L) \cdot (b(n) - M)$ term (the small corner rectangle) is absorbed into $(a(n) - L) \cdot b(n)$, and since $b(n) - M$ is small, this doesn't hurt us. The geometric picture explains why this algebraic trick works: we're decomposing the area difference into manageable pieces, each controlled by making one sequence close to its limit while the other stays bounded.

1.6 Completing the Algebraic Limit Theorem

This result completes the Algebraic Limit Theorem. Combined with earlier results on sums, scalar multiples, and reciprocals of sequences with nonzero limits, we can now compute the limit of any algebraic combination of convergent sequences.

For example, if $a(n) \to L$ and $b(n) \to M$, what is the limit of:

$$\frac{a(n)^2 + 2a(n) + b(n)}{3b(n) + 2 - a(n)^2}$$

Answer: provided the denominator limit is nonzero, the limit is:

$$\frac{L^2 + 2L + M}{3M + 2 - L^2}$$

This is an extremely powerful theorem that allows us to compute limits of complex expressions by simply evaluating the algebraic expression at the limit points.

2 Order Limit Theorem

The Order Limit Theorem establishes a fundamental relationship between inequalities and limits: if a sequence is bounded above (or below) by a constant, then its limit is also bounded by that constant. This result respects the order structure of the real numbers.

This theorem is crucial for establishing inequalities involving limits and is used extensively in analysis to prove comparison theorems, sandwich theorems, and other ordering results.

2.1 New Tools

2.1.1 Definition: SeqBddBy

A sequence $a : \mathbb{N} \to \mathbb{R}$ is **bounded by** M (denoted `SeqBddBy a M`) if $a(n) \leq M$ for all $n \in \mathbb{N}$.

Formally:

$$\text{SeqBddBy}(a, M) \iff \forall n \in \mathbb{N}, a(n) \leq M$$

Note the difference between `SeqBdd` (bounded: $|a(n)| \leq M$) and `SeqBddBy` (bounded above: $a(n) \leq M$).

2.2 The Mathematical Statement

Theorem (OrderLimLe): If $a : \mathbb{N} \to \mathbb{R}$ converges to L and $a(n) \leq K$ for all n, then $L \leq K$.

2.3 Strategic Approach

We prove this by contradiction. Assume $L > K$. Then $L - K > 0$, and we can use this positive quantity as our ε in the definition of convergence.

By convergence, there exists N such that for all $n \geq N$:

$$|a(n) - L| < L - K$$

This means:

$$L - (L - K) < a(n) < L + (L - K)$$

The left inequality gives $a(n) > K$, which contradicts the assumption that $a(n) \leq K$ for all n.

2.4 Lean Solution

```
Statement OrderLimLe (a : ℕ → ℝ) (L : ℝ) (ha : SeqLim a
   L) (K : ℝ) (hK : SeqBddBy a K) :
    L ≤ K := by
by_contra hL
have hL' : K < L := by bound
have hLK : 0 < (L - K) := by linarith [hL']
choose N hN using ha (L - K) hLK
specialize hN N (by bound)
rewrite [abs_lt] at hN
have f1 : L - (L - K) < a N := by linarith [hN.1]
have f2 : K ≤ L - (L - K) := by linarith [hL']
specialize hK N
linarith [f2, hK, f1]
```

2.5 Natural Language Proof

Proof: We proceed by contradiction. Assume $L > K$.

Then $L - K > 0$. Since $a(n) \to L$, by the definition of convergence with $\varepsilon = L - K > 0$, there exists N such that for all $n \geq N$:

$$|a(n) - L| < L - K$$

In particular, taking $n = N$:

$$|a(N) - L| < L - K$$

This is equivalent to:

$$-(L - K) < a(N) - L < L - K$$

From the left inequality:

$$a(N) > L - (L - K) = K$$

But by hypothesis, $a(N) \leq K$ for all N, which contradicts $a(N) > K$.

Therefore, our assumption that $L > K$ must be false, and we conclude $L \leq K$. **QED**

2.6 Importance of Order Preservation

This theorem tells us that limits preserve weak inequalities. Note that strict inequalities are not necessarily preserved: if $a(n) < K$ for all n, we can only conclude $L \leq K$, not $L < K$.

For example, the sequence $a(n) = 1/n$ satisfies $a(n) > 0$ for all n, but $\lim_{n\to\infty} a(n) = 0$, which is not strictly positive.

The Order Limit Theorem is used to prove:

- The Squeeze Theorem (Sandwich Theorem)
- Monotone Convergence Theorem
- Various comparison tests for series
- Properties of suprema and infima of sequences

3 Subsequences

A subsequence is formed by selecting terms from a sequence while preserving their order. There are two equivalent perspectives on what a subsequence is. Most authors refer to $a_{\sigma(n)}$ as a "subsequence of a," viewing it as a new sequence extracted from the original. However, we prefer to call σ itself the subsequence—it is the strictly increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ that tells us which terms to keep. The subsequence of a determined by σ is then the composition $a \circ \sigma$.

Geometrically, imagine the graph of the original sequence $a(n)$. To form a subsequence, you drop out some of the terms and slide everybody over to the left so that every natural number index still has a value. For instance, if you keep only the terms at positions $0, 2, 5, 7, 11, \ldots$, you relabel them as positions $0, 1, 2, 3, 4, \ldots$ in the new sequence. The function σ encodes this relabeling: $\sigma(0) = 0$, $\sigma(1) = 2$, $\sigma(2) = 5$, and so on. The requirement that σ be strictly monotone increasing ($\sigma(i) < \sigma(j)$ whenever $i < j$) ensures that we preserve the original ordering of terms.

The fundamental result is that if a sequence converges, then every subsequence converges to the same limit. This has important consequences: if a sequence has two subsequences converging to different limits, then the sequence itself does not converge.

3.1 New Tools

3.1.1 Definition: Subseq

A function $\sigma : \mathbb{N} \to \mathbb{N}$ is a **subsequence** (denoted `Subseq` σ) if it is strictly increasing:

$$\text{Subseq}(\sigma) \iff \forall i, j \in \mathbb{N}, i < j \implies \sigma(i) < \sigma(j)$$

The subsequence of a determined by σ is the composition $a \circ \sigma : \mathbb{N} \to \mathbb{R}$, where $(a \circ \sigma)(n) = a(\sigma(n))$.

3.1.2 Key Lemma: SubseqGe

If σ is a subsequence, then $n \leq \sigma(n)$ for all n. This can be proved by induction and captures the intuition that we're "spreading out" the indices.

3.2 The Mathematical Statement

Theorem (SubseqConv): If $a : \mathbb{N} \to \mathbb{R}$ converges to L and $\sigma : \mathbb{N} \to \mathbb{N}$ is a subsequence, then $a \circ \sigma$ also converges to L.

3.3 Strategic Approach

The proof is remarkably simple. Given $\varepsilon > 0$, use convergence of a to find N such that $|a(m) - L| < \varepsilon$ for all $m \geq N$.

For any $n \geq N$, we need to show $|(a \circ \sigma)(n) - L| < \varepsilon$, which means $|a(\sigma(n)) - L| < \varepsilon$.

The key observation is that $\sigma(n) \geq n \geq N$ (using the lemma $n \leq \sigma(n)$), so $\sigma(n) \geq N$, and therefore $|a(\sigma(n)) - L| < \varepsilon$ by the convergence of a.

3.4 Lean Solution

```
Statement SubseqConv (a : ℕ → ℝ) (L : ℝ) (ha : SeqLim a
   L) (σ : ℕ → ℕ) (hσ : Subseq σ) :
    SeqLim (a ∘ σ) L := by
intro ε hε
choose N hN using ha ε hε
use N
intro n hn
have f1 : n ≤ σ n := by apply SubseqGe hσ n
have f2 : N ≤ σ n := by linarith [f1, hn]
specialize hN (σ n) f2
apply hN
```

3.5 Natural Language Proof

Proof: Let $\varepsilon > 0$ be given. Since $a(n) \to L$, there exists N such that for all $m \geq N$:

$$|a(m) - L| < \varepsilon$$

We claim that this same N works for the subsequence $a \circ \sigma$. That is, for all $n \geq N$:

$$|(a \circ \sigma)(n) - L| < \varepsilon$$

To see this, let $n \geq N$. By the lemma `SubseqGe`, we have $n \leq \sigma(n)$. Therefore:

$$\sigma(n) \geq n \geq N$$

Since $\sigma(n) \geq N$, we have:

$$|(a \circ \sigma)(n) - L| = |a(\sigma(n)) - L| < \varepsilon$$

This completes the proof. **QED**

3.6 Why This Matters

The Subsequence Theorem has several important consequences:

- If a sequence has two subsequences converging to different limits, then the sequence does not converge. This provides a powerful tool for proving divergence.
- In the Bolzano-Weierstrass theorem, every bounded sequence has a convergent subsequence. If the full sequence converges, it must converge to the same limit as any convergent subsequence.
- Subsequences are used to extract "interesting" behavior from sequences, such as liminf and limsup.
- The concept generalizes to metric spaces and topological spaces, where it plays a crucial role in compactness arguments.

4 Subsequence Example: Oscillating Sequences

We now apply the Subsequence Theorem to a concrete example: the sequence $a(n) = (-1)^n$, which oscillates between -1 and 1. This sequence does not converge, as we previously showed, but it has convergent subsequences.

By extracting the even-indexed terms, we obtain a constant subsequence that converges to 1. Similarly, the odd-indexed terms converge to -1. This example illustrates how subsequences can exhibit simpler behavior than the original sequence.

4.1 New Tools

4.1.1 The let Tactic

The `let` tactic is similar to `have`, but it creates a new named term or function rather than just proving a proposition. This is useful for defining auxiliary variables or functions within a proof.

To create a function in Lean, we use the `fun` keyword (short for "function"). The syntax is:

```
fun x ↦ expression
```

which creates a function that takes input `x` and returns `expression`. The arrow ↦ (typed as `\mapsto`) separates the input variable from the output expression. (If you prefer, you can type `=>` instead of ↦. But I wouldn't.) For example, `fun x ↦ x ^ 2` is the squaring function.

When combined with `let`, we can name and use such functions:

```
let f : ℝ → ℝ := fun x ↦ x ^ 2
```

This creates a function `f` of type $\mathbb{R} \to \mathbb{R}$ (real numbers to real numbers) defined by $f(x) = x^2$.

In our subsequence example, we write:

```
let σ : ℕ → ℕ := fun n ↦ 2 * n
```

This defines $\sigma : \mathbb{N} \to \mathbb{N}$ by $\sigma(n) = 2n$, which selects the even-indexed terms.

4.2 The Mathematical Statement

Theorem: If $a : \mathbb{N} \to \mathbb{R}$ is defined by $a(n) = (-1)^n$, then there exists a subsequence σ and a limit L such that σ is strictly increasing and $a \circ \sigma$ converges to L.

4.3 Strategic Approach

We define $\sigma(n) = 2n$, which picks out the even indices: $0, 2, 4, 6, \ldots$

First, we verify that σ is strictly increasing: if $i < j$, then $2i < 2j$.

Next, we compute $(a \circ \sigma)(n) = a(2n) = (-1)^{2n} = 1$ for all n.

Finally, we show that this constant sequence converges to 1, which is trivial: for any $\varepsilon > 0$, we have $|1 - 1| = 0 < \varepsilon$.

4.4 Lean Solution

```
Statement (a : ℕ → ℝ) (ha : ∀ n, a n = (-1) ^ n) :
    ∃ σ L, Subseq σ ∧ SeqLim (a ∘ σ) L := by
let σ : ℕ → ℕ := fun n ↦ 2 * n
use σ, 1
split_ands
intro i j hij
change 2 * i < 2 * j
linarith [hij]
intro ε hε
use 0
intro n hn
change |a (2 * n) - 1| < ε
specialize ha (2 * n)
rewrite [ha]
have f1 : (-1 : ℝ) ^ (2 * n) = 1 := by bound
rewrite [f1]
norm_num
apply hε
```

4.5 Natural Language Proof

Proof: Define $\sigma : \mathbb{N} \to \mathbb{N}$ by $\sigma(n) = 2n$. We claim that σ is a subsequence and that $a \circ \sigma$ converges to $L = 1$.

Step 1: σ is a subsequence.

Let $i, j \in \mathbb{N}$ with $i < j$. Then:

$$\sigma(i) = 2i < 2j = \sigma(j)$$

Therefore σ is strictly increasing, hence a subsequence.

Step 2: $a \circ \sigma$ converges to 1.
Let $\varepsilon > 0$ be given. We claim that for all $n \geq 0$:

$$|(a \circ \sigma)(n) - 1| < \varepsilon$$

To see this, note that:

$$(a \circ \sigma)(n) = a(\sigma(n)) = a(2n) = (-1)^{2n} = ((-1)^2)^n = 1^n = 1$$

Therefore:

$$|(a \circ \sigma)(n) - 1| = |1 - 1| = 0 < \varepsilon$$

This completes the proof. **QED**

4.6 Implications for Divergence

This example demonstrates that $a(n) = (-1)^n$ does not converge. We can extract subsequences converging to 1 (even indices) and to -1 (odd indices). Since a convergent sequence must have all subsequences converge to the same limit, and we have found subsequences converging to different limits, the sequence a cannot converge.

This technique—showing divergence by exhibiting two subsequences with different limits—is a powerful tool in analysis.

Objects:

A B C L

n : N

Assum —

Goal: ———

Induct on n.

Objects

A B C

Assum.

Goal: $\exists L, \forall n \in N, \ldots$

Induct on n.

Goal: $\exists K, \forall n, n(n+1)(2n+1) = 6K$.

$\Rightarrow$ Goal: $\forall n, \exists K, \ldots$

N.

Goal: $\forall n < N, \ldots$

~~Induct on~~ hn

Goal: N n : N

hn : $n < N$

Goal: ——— n+1 = Succ n.

"Principle": If property does not continue to hold for Succ n, then that is not a good variable for induction.

Thm: $a_n \to L \neq 0$ (ha) (hL), $b_n \to M \neq 0$ (hb) (hm), $c_n = a_n \cdot b_n$. (hc)

Goal: $c_n \to L \cdot M$.

Sketch: Let $\varepsilon > 0$ (hε) be given.

Choose K hK using B dd of Conv b M hb hM.

$b \geq |L|$

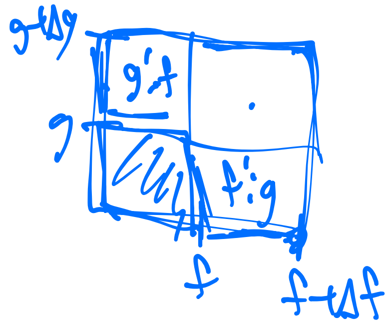
$0 < K$

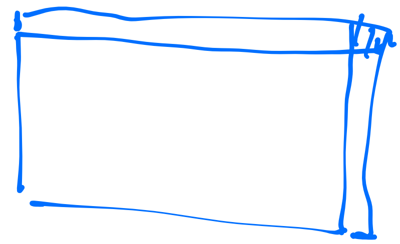
$$|c_n - L\cdot M| = |a_n\cdot b_n - L\cdot M| = \boxed{|a_n b_n - b_n L + b_n L - L\cdot M|}$$

$$= |b_n(a_n - L) + L(b_n - M)|. \quad \text{(abs\_add)}$$

$$\le \underbrace{|b_n|}_{\le K}\underbrace{|a_n - L|}_{< \frac{\varepsilon}{2\cdot K}} + |L|\cdot\underbrace{|b_n - M|}_{< \frac{\varepsilon}{2|L|}} \quad \text{(abs\_mul)}.$$

$$(f\cdot g)' = f'\cdot g + f\cdot g'.$$





$a_n \to L$, $b_n \to M$

Recall
1. $a_n \to L$, Goal: $c\cdot a_n \to c\cdot L$. ✓
2. Goal: $a_n + b_n \to L + M$ ✓
3. $a_n \to L \neq 0$ Goal: $1/a_n \to 1/L$ ✓
4. $a_n\cdot b_n \to L\cdot M$. ✓

Corollary: $a_n \to L$, $b_n \to M$,

$$\frac{a_n^2 + 2a_n + b_n}{3b_n + 2 - a_n^2} \to \frac{L^2 + 2L + M}{\underbrace{3M + 2 - L^2}_{\neq 0}}.$$

Any rational function commutes with limits.

Order Limit Thm: $a_n \to L$, ha.

hK: ~~$\forall n,\ a_n \le K$~~ SeqBddBy a K   Goal: $L \le K$.

Not.

a : $\mathbb{N} \to \mathbb{R}$

Compare: $a \le K$

by_contra h   h : $\neg L \le K$,

have h': $0 < L - K$:= by done.

choose N hN using ha (L-K) h',

specialize hN N (by done)   N : $\mathbb{N}$

rewrite [abs_lt] at hN   hN: $\forall n \ge N, |a\, n - L| < L-K$,

specialize hK N   hN: $|a\, N - L| < L-K$

$(\max\{k, N\})$.

$n \geq N: (L-K) < a_N - L$
$\wedge \; a_N - L < K$

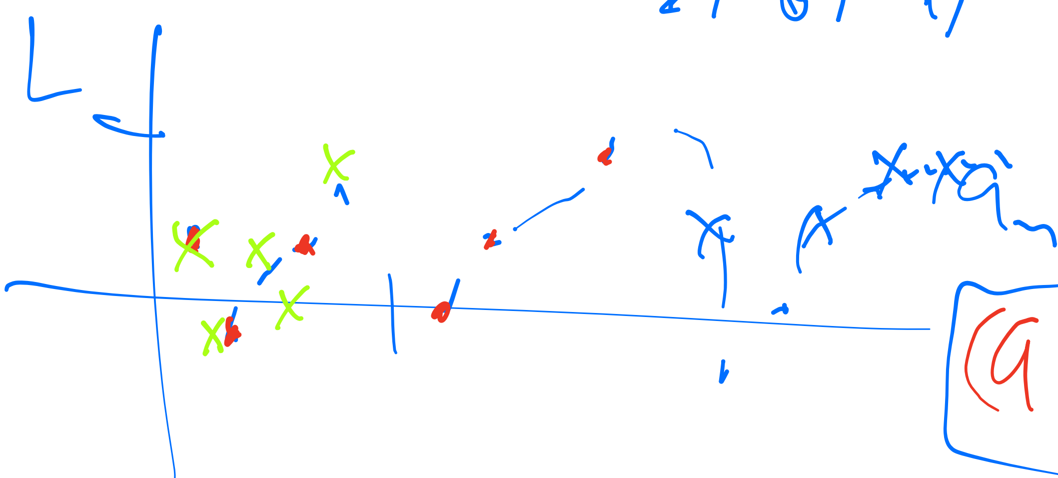
Def: $\sigma : \mathbb{N} \to \mathbb{N}$ is a subsequence if $i < j \Rightarrow \sigma_i < \sigma_j$.

a_{n_k}

"monotonic strictly ($\neq$) increasing"

$a_0, a_1, a_2, a_3 \text{ —— }$

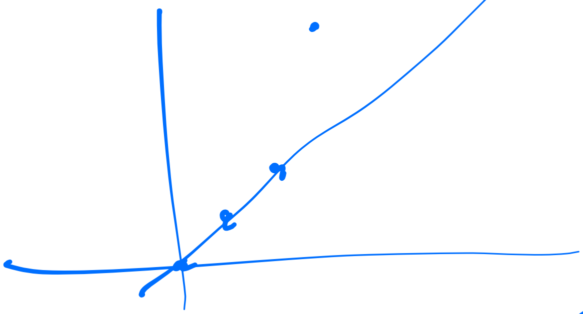
$a_0, a_3, a_6, a_9, \ldots$



$\boxed{(a \circ \sigma)_n} = a_{\sigma_n}$.

Thm (Subseq Limit): $a_n \to L$ & σ = subseq. Goal: $a_{\sigma_n} \to L$.

Recall: (Pset 8 #4: σ subseq $\Rightarrow \sigma(n) \geq n$)



SubseqGe.

Goal: $\exists N, \forall n \geq N,$
$|a_{\sigma(n)} - L| < \varepsilon.$

a_{σ_n}

intro ε hε.

Choose N hN using hε.

Use $\sigma(N)$.

intro n hn